# Limit Theorems for Critical Binary Branching Age-dependent Particle Systems with Heavy-tailed Lifetimes

by

Antonio Murillo Salas

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

**Doctor of Philosophy**

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*To my mother*

*and to the memory of*

*my grandparents Antonio and Refugio*

# Contents

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# Introduction

Branching particle systems constitute a vigorous area of contemporary theory of stochastic processes. From the theoretical point of view, this kind of models is relevant because of the rich mathematical structure associated to their qualitative behavior, and of their connection with other areas of mathematics, such as partial differential equations and analysis. Also, but not less important, branching particle system are very useful in applications in fields like biology, genetics, statistical physics, ecology, epidemics, etc.

In this thesis we study a branching system of particles living in $d$-dimensional Euclidean space $\mathbb{R}^d$, which can be described roughly as follows. Individuals or particles are subject to random motions, random lifetimes and random branching. More precisely, we study a critical binary branching population in which each particle, during a random lifetime $\tau$, performs a spherically symmetric $\alpha$-stable process in $\mathbb{R}^d$, $\alpha \in (0, 2]$. At the end of the particle's lifetime either it branches into two new particles with probability 1/2, and in this case the new particles appear at the position where their progenitor died, or the particle disappears with probability 1/2. We assume that $\tau$ has a distribution function belonging to the domain of attraction of a $\gamma$-stable law, with $\gamma \in (0, 1)$. More specifically, we assume hat $\tau$ is a non-arithmetic random variable and possesses a distribution function $F$ such that $\text{supp}(F) \subset [0, \infty)$, $F(0) = 0$, $F(x) < 1$ for all $x \in [0, \infty)$, and

$$1 - F(u) \sim u^{-\gamma}/\Gamma(1-\gamma) \quad \text{a}s \quad u \to \infty \tag{1}$$

for some $\gamma \in (0, 1)$, where $\Gamma(\cdot)$ denotes the gamma function. Also, we assume that the population starts off from a Poisson random field on $\mathbb{R}^d$ with intensity measure $\Lambda$, $\Lambda$ being the Lebesgue measure on $\mathbb{R}^d$. Finally, we suppose all the standard independence assumptions in branching systems. We write $X_t(A) \equiv X(t, A)$ for the number of individuals living in the Borel set $A \subset \mathbb{R}^d$ at time $t \geq 0$. Notice that the process $X \equiv \{X_t, t \geq 0\}$ takes values in the space of locally finite counting measures on $\mathbb{R}^d$. A similar model, but with a more general branching mechanism, has been studied by Kaj and Sagitov (1998), Vatutin and Wakolbinger (1999) and Fleischmann et. al. (2002).

Before we go into the details of the kind of results obtained in this work, we give a brief discussion of some, by now classical results, in the case of exponentially distributed lifetimes.

Consider the branching model described above but with exponentially distributed lifetimes, i.e., $F(t) = 1 - e^{ct}$, $t \geq 0$, for some constant $c > 0$. Throughout this work such model will be referred to as *critical binary branching system*. An important consequence of the assumption of exponentially distributed lifetimes is that the measure-valued process $\{X_t, t \geq 0\}$ enjoys the Markov property. The model we shall study in this thesis, having non-exponential lifetimes, will be called *critical binary age-dependent branching system* and, in general, the process $\{X_t, t \geq 0\}$ is not Markovian. Kaj and Sagitov (1998) and Fleischmann et. al. (2002) recover the Markov property of their models by enlarging the phase-space of the population, attaching to each particle its residual life time.

Dawson (1977) and Dawson and Ivanoff (1978) investigated the asymptotic behavior (extinction vs. persistence) of the critical binary branching system. They proved that this process becomes extinct if $d \leq \alpha$, namely, $X_t \xrightarrow{v} 0$ in probability as $t \longrightarrow \infty$, where $\xrightarrow{v}$ denotes convergence in the vague topology. On the other hand, if $d > \alpha$, $X_t \xrightarrow{v} X_\infty$ in distribution as $t \longrightarrow \infty$, where $X_\infty$ is an equilibrium state with $\mathbb{E}X_\infty = \Lambda$ (this is called persistence), see Gorostiza and Wakolbinger (1991). Gorostiza (1983) studied the so-called high density fluctuation limit of this branching system (under a more general branching mechanism). The multi-type version of this model was studied by López-Mimbela (1992). Vatutin and Wakolbinger (1999) investigated the persistence vs.

extinction dichotomy of an age-dependent critical branching system, where the offspring number $\zeta$ of any individual has probability generating function given by

$$\Phi(s) := \mathbb{E}s^{\zeta} = s + c(1-s)^{1+\beta}, \tag{2}$$

with $|s| \leq 1$, $\beta \in (0,1]$ and $c \in (0, \frac{1}{1+\beta}]$. Notice that the choice $\beta = 1$ and $c = 1/2$ gives the critical binary branching case. This branching law will be referred as $(1+\beta)$-branching. Vatutin and Wakolbinger (1999) have shown the following: (a) Assume that $\tau$ is a non-arithmetic random variable such that $\mathbb{E}\tau < \infty$, and that the initial population is Poisson distributed with intensity measure $\Lambda$. Then the population is persistent for $d > \alpha/\beta$, and goes to local extinction when $d \leq \alpha/\beta$. Hence, if $\mathbb{E}\tau < \infty$, the critical dimension for persistence is the same as for the branching system with exponentially distributed lifetimes. (b) If $\tau$ has heavy tail as in (1), then the critical dimension is $d = \alpha\gamma/\beta$, meaning that the population is persistent for $d > \alpha\gamma/\beta$ and goes to local extinction for $d < \alpha\gamma/\beta$. Later on, Fleischmann et. al. (2002) proved that persistence also holds at the critical dimension. By Markovianizing the model through the residual lifetime of each particle, Kaj and Sagitov (1998) have shown that the age-dependent branching particle system admits a diffusion-type approximation, i.e., a limit procedure similar to the one used to obtain the so-called Dawson-Watanabe superprocess. Moreover, Fleischmann et. al. (2002) investigated scaling properties of the diffusion limit and absolutely continuity properties of its states.

From the above paragraph one learns that, contrary to the case of finite-mean lifetimes, the dimension for persistence in presence of heavy-tailed life times changes according to the decay exponent of the tail. In the case of finite variance branching, this reveals that there is a kind of competition (or compensation) between longevity of individuals and the transience/recurrence property of the motion process. Heavy-tailed lifetimes enhance the mobility of individuals, favoring the spreading out of the the particles and thus counteracting the tendency to local extinction, see Vatutin and Wakolbinger (1999) and Fleischmann et. al. (2002). Hence, it would be interesting to investigate further properties of this model, which could display more differences with respect to the exponentially distributed lifetimes case, or even with the case of lifetimes with finite mean. Besides, there is no genuine reason to assume exponentially distributed lifetimes in our model, and in applications it is more realistic to assume a more general non-exponential lifetime distribution.

The occupation time process of a branching system is another object that has been extensively studied in the context of exponentially distributed lifetimes (see Cox and Griffeath (1985), Méléard and Roelly (1992), Bojdecki et. al. (2006a), Bojdecki et. al. (2006b)). See also Bojdecki et. al. (2007b) for the case of $(1+\beta)$-branching. Iscoe (1986a) and Fleischmann and Gärtner (1986) investigated the occupation time of Dawson-Watanabe superprocesses, i.e., measure-valued processes which are diffusion limits of branching particle systems with exponential lifetimes.

The *occupation time process* of a *càdlàg* measure-valued process $Y \equiv \{Y_t,\, t \geq 0\}$, is again a measure-valued process $J \equiv \{J_t, t \geq 0\}$ which is defined by

$$\langle \psi, J_t \rangle := \int_0^t \langle \psi, Y_s \rangle ds, \quad t \geq 0,$$

for all bounded measurable function $\psi : \mathbb{R}^d \to \mathbb{R}_+$, where $\langle \psi, \mu \rangle \equiv \int \psi d\mu$ with $\mu$ a measure on $\mathbb{R}^d$. Cox and Griffeath (1985) and Méléard and Roelly (1992) proved a strong law of large numbers for the occupation time of a critical binary branching system. Namely, as $t \longrightarrow \infty$,

$$t^{-1} \langle \psi, J_t \rangle \overset{a.s.}{\to} \langle \psi, \Lambda \rangle$$

for all positive continuous function $\psi$ with compact support. Moreover, Cox and Griffeath (1985) proved the following central limit-type theorem for the occupation time of the critical binary branching Brownian motion: As $t \to \infty$,

$$\frac{\langle J_t, 1_A \rangle - \langle 1_A, \Lambda \rangle t}{b_t} \Rightarrow \mathfrak{N}(0, \sigma^2), \quad A \in \mathcal{B}(\mathbb{R}^d),$$

where $b_t$ is a function that depends on the spatial dimension $d$, and $\sigma^2$ is a positive constant that also may vary with the dimension. The dimension dependence of the normalizing function $b_t$ is a typical characteristic in this theory because the behavior of the branching system is highly related to the transience/recurrence behavior of the motion process.

Bojdecki et. al. (2006a) and Bojdecki et. al. (2006b) have investigated the limit fluctuations of the rescaled occupation time process of this branching system. They have shown that in case of "low dimensions", $\alpha < d < 2\alpha$, these limits are processes which exhibit long-range dependence behavior, such as *fractional Brownian motion* and *sub-fractional Brownian motion*. See also Birkner and Zähle (2007) for related results, where the underlying process is a branching random walk in the $d$-dimensional lattice. See Bojdecki et. al. (2007b) for the case of $(1+\beta)$-branching with $\beta < 1$, where some long-range dependence self-similar non Gaussian process appear in the case of intermediate dimensions $\alpha/\beta < d < \alpha(1+\beta)/\beta$. Talarczyk (2007) studied a functional ergodic result at the critical dimension, $d = \alpha/\beta$, in the case of $(1+\beta)$-branching. See Milos (2007) for a general critical finite variance branching law in a population starting off either from a standard Poisson random field or from the equilibrium distribution for intermediate dimensions ($\alpha < d < 2\alpha$).

Iscoe (1986a) proved a central limit-type theorem for the occupation time of the Dawson-Watanabe superprocess; see also Fleischmann and Gärtner (1986) for more results in this direction.

Our aim in this thesis is to perform a step forward, by investigating some of the above-mentioned limit results and properties in the case of critical binary-branching age-dependent particle system. In what follows we give an outline of this work, as well as an (informal) description of the obtained results.

We start in Chapter 1 with a brief review of background results that we need to develop this work. Proofs are omitted, but we provide references where they can be found.

Chapter 2 contains the precise definition of the model studied in this work. Also, some moments calculations which we shall use in the subsequent chapters are given there.

In Chapter 3 we investigate the so-called *high density* and *space-time scaling* limits of our age-dependent branching system. The high density limit consist in increasing the initial intensity by a factor $K$ which will tend to infinity, see Martin-Löf (1976) for the physical motivation of this rescaling. We are interested in the fluctuations process, i.e, we center the process around its mean measure and normalize it by $K^{1/2}$; this entails to change the state-space of $X$ and the underlying notion of convergence. We show that the fluctuations process converges to an $S'(\mathbb{R}^d)$-valued centered Gaussian process whose covariance functional is calculated explicitly, where $S'(\mathbb{R}^d)$ denotes the strong dual of the space $S(\mathbb{R}^d)$ of rapidly decreasing functions, see Section 1.2 in Chapter 1. Also we prove several properties of the limit process, namely, Markov property, almost sure continuity of paths in the norm $\|\cdot\|_{-p}$ (which is a norm on a subspace of $S'(\mathbb{R}^d)$ which renders a stronger topology than that of $S'(\mathbb{R}^d)$) for some $p \geq 1$, and the form of the spectral measure. These results are valid for a general non-arithmetic lifetime distribution. When the lifetime distribution possesses a continuous density, we also show that the limit process satisfies a generalized Langevin equation. These results were known only in the case of exponentially distributed lifetimes; see Gorostiza (1983) for the general mono-type branching case, and López-Mimbela (1992) for systems with multi-type branching.

For the space-time scaling limit we assume that the lifetime distribution has a tail of the form (1). The coordinates in space and time are respectively $Kx$ and $K^{\alpha}t$, again $K$ being a parameter which will tend to infinity. As in the high density limit, we are interested in the asymptotic normalized fluctuations of the process. In this case we need to assume that $d > \alpha\gamma$, i.e., we require supercritical dimension for persistence. The normalizing constant for the fluctuation process is $K^{d+\alpha\gamma}$, with $K \longrightarrow \infty$. (Recall that, for exponentially distributed lifetimes, the normalizing function is $K^{d+\alpha}$; see Gorostiza (1983)). The limit process is again an $S'(\mathbb{R}^d)$-valued centered Gaussian process, it is a Markov process and possesses a version which is continuous in the norm $\|\cdot\|_{-p}$ for some $p \geq 1$. Also, it satisfies a generalized Langevin equation. Heavy-tailed lifetimes play a key role in the space-time scaling because the power $\gamma$ of the tail decay figures explicitly in the limit theorems, the effect being similar to the one that it has in the diffusion limit approximation of Kaj and Sagitov (1998); see equation (5.1) there.

It is well known that, in order to prove weak convergence of a sequence $\{P_n\}_{n=1}^{\infty}$ of probability measures in the Skorokhod space, it is sufficient to show weak convergence of the finite-dimensional marginals, and tightness (or relative compactness) of $\{P_n\}$. In our proof of the fluctuation limit theorems mentioned above, convergence of finite-dimensional distributions is achieved by the usual method, showing convergence of characteristic functionals and using the Minlos-Sasonov theorem. The proof of tightness can not be carried out as in the classical case of exponentially distributed life times because, as we mentioned above, $\{X_t,\, t \geq 0\}$ is not a Markov process, and many of the steps in the proof of tightness are based on this property. To overcome this difficulty, we consider the Markov process $\{X_t \times \bar{X}_t,\, t \geq 0\}$, where $\{\bar{X}_t,\, t \geq 0\}$ is a Markovianization of the branching system $\{X_t,\, t \geq 0\}$ obtained by enlarging the phase-state, including the "elapsed time" or "age" of each individual (see Appendix A for a more detailed discussion, and see Kaj and Sagitov (1998) and Fleischmann et. al. (2002) for a related procedure based on the residual lifetime of each particle).

Chapter 4 is devoted to prove laws of large numbers for the occupation time process. We prove that in supercritical dimensions, namely $d > \alpha\gamma$ for lifetime distributions satisfying (1), and $d > \alpha$ for finite-mean lifetimes, the occupation time of the critical binary age-dependent branching system satisfies a strong law of large numbers. This result is similar to that obtained by Cox and Griffeath (1985) and Méléard and Roelly (1992) in case of exponentially distributed lifetimes.

In case of heavy-tailed lifetimes the proof of the strong law of large numbers is carried out into two steps. In the first step we consider the case of "low" dimensions $\alpha\gamma < d < 2\alpha$, and in the second step we deal with "large" dimensions $d \geq 2\alpha$. In the case of low dimensions we consider a general non-arithmetic lifetime distribution with tail of the form (1). We use the covariance functional of the non-Markovian branching system described in Chapter 2, and certain subtle techniques from Bojdecki et. al. (2006a) and Bojdecki et. al. (2006b), which we adapted to our non-Markovian scenario. We were unable to extend this method to all dimensions because of the lack of proper upper-bounds for the variance functional of the re-scaled occupation time.

To deal with the case of large dimensions, we use the Markovianized branching system introduced in Appendix A. This allows us to use directly the well-known self-similarity of $\alpha$-stable transition densities, as was done in Méléard and Roelly (1992). Here we must say that, in order to use this procedure, it is necessary to assume that the lifetime distribution possesses a continuous density function. This contrasts with the case of low dimensions, where no absolute continuity condition is required. We think, however, that the result should be true for general lifetime distributions.

In the case of a general non-arithmetic lifetime distribution having finite mean, we show that the strong law of large numbers holds for all $d > \alpha$. Notice that this conclusion extends the known result of Cox and Griffeath (1985) and Méléard and Roelly (1992) in the case of exponentially distributed lifetimes. The proof is carried out using estimates of the variance functional of the occupation time process, as well as bounds of $\alpha$-stable distributions.

Up to now, we have investigated fluctuation limit theorems for our branching model under various re-scalings, and strong laws of large numbers of the occupation time process for a wide class of lifetime distributions. Our next goal is to go further by investigating fluctuations of the occupation time process. In Chapter 5, we present some work in progress regarding the occupation time fluctuation process and some comments for future work.

Occupation time fluctuations of branching systems have been extensively investigated in recent years by T. Bojdecki, L.G. Gorostiza and A. Talarczyc, in the case of exponentially distributed lifetimes. As we mentioned earlier, these authors obtained fluctuation limits which are Gaussian processes exhibiting long-range dependence behavior, such as fractional Brownian motion or sub-fractional Brownian motion. See Bojdecki et. al.

(2007b) where other non Gaussian long-range dependence processes appear.

For our model, we have shown that the covariance function of the re-scaled occupation time process converges in dimensions $\alpha\gamma < d < 2\alpha$ to a covariance function $C(\gamma)$ depending on $\gamma$. Moreover, $C(1)$ (which corresponds to lifetimes with finite mean) is proportional to the covariance function of the sub-fractional Brownian motion. This insinuates that, regardless of the exponential -or otherwise- distribution lifetime, as soon as $E\tau < \infty$, the occupation time fluctuations should have a distribution of the type of sub-fractional Brownian motion, or of one of its relatives. Although convergence of covariance functions does not imply any kind of convergence of the underlying processes, this suggests a further research to show that, at least, weak convergence of finite-dimensional distributions holds. Such a result would extend Theorem 2.2 of Bojdecki et. al. (2006a) to any lifetime distribution with finite mean. In case of heavy-tailed lifetimes, the form of the limiting covariance function suggests the existence of a new Gaussian process with long-range dependence behavior.

# Chapter 1

# Preliminaries

## 1.1 The space $\mathcal{M}_p(\mathbb{R}^d)$

For a more detailed discussion on the following topic see Ethier and Kurtz (1986) and Kallenberg (1983).

Let $\mathbb{R}^d$ be the $d$-dimensional Euclidean space and $|\cdot|$ the usual norm on it. Let $C(\mathbb{R}^d)$ be the space of continuous real-valued functions $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}$. $C_c(\mathbb{R}^d)$ denotes the subspace of continuous functions with compact support, and let $C_c(\mathbb{R}^d)_+$ denote the subset of non-negative elements of $C_c(\mathbb{R}^d)$. Let $C_0(\mathbb{R}^d)$ be the set of elements of $C(\mathbb{R}^d)$ vanishing at infinity.For each $p \geq 0$ we define the reference function

$$\phi_p(x) = (1 + |x|^2)^{-p}, \;\; x \in \mathbb{R}^d.$$

We denote by $\mathcal{M}_p(\mathbb{R}^d)$ the space of non-negative Radon measures $\mu$ on $\mathbb{R}^d$, such that $\int \phi_p d\mu < \infty$, and endow $\mathcal{M}_p(\mathbb{R}^d)$ with the $p$-vague topology, i.e., the minimal topology under which the maps $\mu \longmapsto \int \phi d\mu$ are continuous for $\phi \in K_p(\mathbb{R}^d)_+$, where $K_p(\mathbb{R}^d)_+ = C_c(\mathbb{R}^d)_+ \cup \{\phi_p\}$.

$\mathcal{M}_p(\mathbb{R}^d)$ is a complete, separable metric space, and the finite atomic measures are dense in it. We denote by $\mathcal{N}_p(\mathbb{R}^d) \subset \mathcal{M}_p(\mathbb{R}^d)$ the subspace of counting measures. The Lebesgue measure on $\mathbb{R}^d$ belongs to $\mathcal{M}_p(\mathbb{R}^d)$ for $p > d/2$. $D(\mathbb{R}_+, \mathcal{M}_p(\mathbb{R}^d))$ denotes the space of functions from $\mathbb{R}_+$ to $\mathcal{M}_p(\mathbb{R}^d)$ which are right continuous left limited. $D(\mathbb{R}_+, \mathcal{M}_p(\mathbb{R}^d))$ equipped with the Skorokhod topology is a complete, separable metric space (Ethier and Kurtz (1986)).

Existence of versions of processes in $D(\mathbb{R}_+, \mathcal{M}_p(\mathbb{R}^d))$ can not be deduced directly from usual results, because $\mathcal{M}_p(\mathbb{R}^d)$ with the $p$-vague topology is not locally compact. However, $\mathcal{M}_p(\mathbb{R}^d)$ can be seen as a subset of the locally compact space $\mathcal{M}_p(\dot{\mathbb{R}}^d)$, which is defined in the following paragraph.

A complete discussion on the following can be found in Iscoe (1986a), Dawson and Gorostiza (1990) and references there in.

Let $\Xi$ be an isolated point and define $\dot{\mathbb{R}}^d = \mathbb{R}^d \cup \{\Xi\}$. Now, for each $p \geq 0$ define $\dot{\phi}_p$ on $\dot{\mathbb{R}}^d$ as follows,

$$\dot{\phi}_p(x) = \begin{cases} \phi_p(x) & \text{if } x \in \mathbb{R}^d, \\ 1 & \text{if } x = \Xi. \end{cases}$$

The space $\mathcal{M}_p(\dot{\mathbb{R}}^d)$ is the set of non-negative Radon measures $\mu$ on $\dot{\mathbb{R}}^d$ such that,

$$\int \phi_p d\mu|_{\mathbb{R}^d} + \mu(\{\Xi\}) < \infty,$$

equipped with the $p$-vague topology, which is defined analogously as in $\mathcal{M}_p(\mathbb{R}^d)$, where the functions $\phi$ are in

$$K_p^\infty(\dot{\mathbb{R}}^d) = C_c^\infty(\mathbb{R}^d) \cup \{\dot{\phi}_p\},$$

where $C_c^\infty(\mathbb{R}^d)$ denotes the space of infinitely differentiable functions $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}$ with compact support. Let $C_p(\mathbb{R}^d)$ be the space of real-valued continuous functions on $\mathbb{R}^d$ such that

$$|\phi|_p := \sup_{x \in \mathbb{R}^d} |\phi(x)/\phi_p(x)| < \infty,$$

and

$$C_{p,0}(\mathbb{R}^d) := \left\{ \phi \in C(\mathbb{R}^d) : \phi/\phi_p \in C_0(\mathbb{R}^d) \right\}$$

Then $C_p(\mathbb{R}^d)$ and $C_{p,0}(\mathbb{R}^d)$ are Banach spaces with respect to the norm $|\cdot|_p$. Let $C_p(\dot{\mathbb{R}}^d)$

be the space of real-valued continuous functions $\phi$ on $\dot{\mathbb{R}}^d$ such that $\phi(\Xi) = c \in \mathbb{R}_+$ and

$$\lim_{|x|\to\infty} |x|^{2p}|\phi(x)| = c.$$

Note that $K_p(\mathbb{R}^d) \subset C_p(\mathbb{R}^d)$ and $K_p(\dot{\mathbb{R}}^d) \subset C_p(\dot{\mathbb{R}}^d)$.

Finally, we introduce the following notation, if $\mu$ is a measure and $\phi$ a $\mu$-integrable function

$$\langle \phi, \mu \rangle := \int \phi d\mu.$$

## 1.2 Schwartz spaces

Let $S(\mathbb{R}^d)$ be the space of rapidly decreasing functions, i.e. functions $\phi : \mathbb{R}^d \to \mathbb{R}$ such that $\phi$ is infinitely differentiable, and for all $p = 0, 1, 2, \cdots,$

$$\|\phi\|_p = \left( \sum_{|k|=0}^{p} \int_{\mathbb{R}^d} \left(1 + |x|^2\right)^p |D^k\phi(x)|^2 dx \right)^{1/2} < \infty, \tag{1.1}$$

where $x = (x_1, \cdots, x_d)$, $k = (k_1, \cdots, k_d)$, $|k| = k_1 + \cdots + k_d$ and $D^k = \partial^{|k|}/\partial x_1^{k_1} \cdots \partial x_d^{k_d}$. It can be shown that $S(\mathbb{R}^d) \subset C_p(\mathbb{R}^d)$.

The space $S(\mathbb{R}^d)$ endowed with the topology induced by the system of Hilbert's norms $\{\|\cdot\|_p, p \geq 0\}$ is a metric space which is separable, complete and nuclear. Let $S_p(\mathbb{R}^d)$ be completion of $S(\mathbb{R}^d)$ with respect to the norm $\|\cdot\|_p$. Then $S_m(\mathbb{R}^d) \subset S_n(\mathbb{R}^d)$ for $n \leq m$, $S(\mathbb{R}^d) = \cap_{p\geq 0} S_p(\mathbb{R}^d)$, and for each $p \geq 0$, $S_p(\mathbb{R}^d)$ is a Hilbert space. In particular, $S_0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$. Let us denote by $S_p'(\mathbb{R}^d)$ and $S'(\mathbb{R}^d)$ the strong dual space of $S_p(\mathbb{R}^d)$ and $S(\mathbb{R}^d)$, respectively. $S'(\mathbb{R}^d)$ is nuclear and is called the *Schwartz's space of tempered distributions* on $\mathbb{R}^d$.

For each $p = 0, 1, 2, \cdots$, $S_p'(\mathbb{R}^d)$ is a Hilbert space with norm

$$\|F\|_{-p} := \sup_{\|\phi\|_p=1} |\langle F, \phi \rangle|, \quad F \in S_p'(\mathbb{R}^d), \quad \phi \in S_p(\mathbb{R}^d), \tag{1.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the canonic bilinear form in $S'(\mathbb{R}^d) \times S(\mathbb{R}^d)$ and $S_p'(\mathbb{R}^d) \times S_p(\mathbb{R}^d)$. We denote by $D(\mathbb{R}_+, S'(\mathbb{R}^d))$ the space of functions from $\mathbb{R}_+$ to $S'(\mathbb{R}^d)$ which are continuous from the right with limits from the left, endowed with the Skorokhod topology (see Mitoma (1983b)).

For more details on this topic see Gelfand and Vilenkin (1966) and Treves (1967).

## 1.3 The spherically symmetric $\alpha$-stable process

Let $B \equiv \{B_t, t \geq 0\}$ be the spherically symmetric stable process in $\mathbb{R}^d$, with index $\alpha \in (0, 2]$. Then $B$ is a homogeneous strong Markov process with transition functions $\{p_t^\alpha, t > 0\}$ given by

$$\begin{aligned} p_t^\alpha(x) &= p_t^\alpha(\cdot, \cdot + x) \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp\left(-ix \cdot y - t|y|^\alpha\right) dy, \quad t > 0, \quad x \in \mathbb{R}^d, \end{aligned} \tag{1.3}$$

where $x \cdot y$ denotes the scalar product on $\mathbb{R}^d$. The case $\alpha = 2$ corresponds to Brownian motion in $\mathbb{R}^d$ with variance parameter 2.

In what follows some of the constants used can change from line to line; we will indicate the place of definition of constants by subscripts.

Let $\{\mathcal{S}_t, t \geq 0\}$ be the semigroup of operators on $L^2(\mathbb{R}^d)$ associated to the process $B$, i.e. for each $t \geq 0$,

$$\begin{aligned} (\mathcal{S}_t\varphi)(x) &:= \mathbb{E}\left[\varphi(B_t) | B_0 = x\right] \\ &= \int_{\mathbb{R}^d} p_t^\alpha(x - y)\varphi(y) dy, \quad x \in \mathbb{R}^d, \quad \varphi \in L^2(\mathbb{R}^d). \end{aligned}$$

For any $\psi \in L^2(\mathbb{R}^d)$ and $K \geq 1$ we denote by $\psi^K$ the map $x \longmapsto \psi(x/K)$, $x \in \mathbb{R}^d$. It can be seen from (1.3) that the self-similarity property

$$\mathcal{S}_t \psi^K = (\mathcal{S}_{t/K^\alpha} \psi)^K \tag{1.4}$$

holds for all $t \geq 0$ and $K \geq 1$. By self-similarity and unimodality of stable densities we also have

$$p_t^\alpha(x) \leq c_{(1.5)} t^{-d/\alpha}, \quad t > 0, \quad x \in \mathbb{R}^d, \tag{1.5}$$

where $c_{(1.5)}$ is a positive constant. Also, the bound

$$p_t^\alpha(x) \leq c_{(1.6)} t|x|^{-d-\alpha}, \quad t > 0, \quad x \in \mathbb{R}^d, \tag{1.6}$$

holds for some positive constant $c_{(1.6)}$, see Fleischmann and Gärtner (1986). The infinitesimal generator of $\{\mathcal{S}_t, t \geq 0\}$ is denoted by $\Delta_\alpha$ and is given by $\Delta_\alpha := -(-\Delta)^{\alpha/2}$, where $\Delta$ in the Laplacian operator defined on $C_c^\infty(\mathbb{R}^d)$ (see Sato (1999), Chapter 6); it is known that $S(\mathbb{R}^d) \subset \text{Dom}(\Delta_\alpha)$. When $\alpha = 2$, $\Delta_2 = \Delta$ and $\{\mathcal{S}_t, t \geq 0\}$ are linear operators from $S(\mathbb{R}^d)$ into itself. This is not the case when $\alpha < 2$, and in this case it is better to work with the space $C_{p,0}(\mathbb{R}^d)$, $p > 0$.

**Theorem 1.3.1** *(Dawson and Gorostiza (1990)) For $\alpha < 2$ and $d/2 < p < (d+\alpha)/2$ the following inclusions holds*

$$S(\mathbb{R}^d) \subset C_{p,0}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset C'_{p,0}(\mathbb{R}^d) \subset S'(\mathbb{R}^d),$$

*$S(\mathbb{R}^d)$ is continuously and densely embedded in $C_{p,0}(\mathbb{R}^d)$. For any $t \geq 0$, $\mathcal{S}_t$ is a continuous linear operator from $S(\mathbb{R}^d)$ into $C_{p,0}(\mathbb{R}^d)$. Moreover $\Delta_\alpha$ transforms continuously $S(\mathbb{R}^d)$ into $C_{p,0}(\mathbb{R}^d)$.*

## 1.4 Gaussian processes

In this Section we introduce the notion of *Gaussian process* with values in $S'(\mathbb{R}^d)$. Through this section all random elements will be defined on a complete probability space $(\Omega, \mathcal{F}, P)$. We recall some results regarding $S'(\mathbb{R}^d)$-valued Gaussian processes. Proofs and references can be found in Fernández (1986). If $E$ is a topological space, $\mathcal{B}(E)$ denotes the Borel $\sigma$-field in $E$.

**Definition 1.4.1** *Let $X$ be a random variable with values in $S'(\mathbb{R}^d)$. The functional*

$$\hat{F}_X(\phi) = \mathbb{E}\left[e^{i\langle \phi, X\rangle}\right], \quad \phi \in S(\mathbb{R}^d),$$

*is called the characteristic functional of $X$. We say that $X$ is Gaussian if its characteristic functional is given by*

$$\hat{F}_X(\phi) = \exp\left(i\mu(\phi) - \frac{1}{2}K(\phi,\phi)\right), \quad \phi \in S(\mathbb{R}^d),$$

*where $\mu(\phi)$ is a continuous linear functional on $S(\mathbb{R}^d)$ and $K(\phi,\psi)$ is a positive definite bilinear form on $S(\mathbb{R}^d)\times S(\mathbb{R}^d)$. The functionals $\mu$ and $K$ are called mean and covariance functional of $X$, respectively.*

**Theorem 1.4.2** *(Itô (1984)) Let $X$ be an $S'(\mathbb{R}^d)$-valued random variable. Then $\hat{F}_X$ satisfies the following properties:*

1. *$\hat{F}_X$ is positive definite, i.e., for all $n \in \mathbb{N}$, $a_k \in \mathbb{C}$ and $\phi_j \in S(\mathbb{R}^d)$, $k,j = 1,2,\cdots,n$,*
$$\sum_{j,k=1}^{n} a_j \bar{a}_k \hat{F}_X(\phi_j - \phi_k) \geq 0.$$
2. *$\hat{F}_X(0) = 1$.*
3. *$\hat{F}_X(\phi)$ is continuous at $\phi = 0$.*

**Theorem 1.4.3** *(Bochner-Minlos, Itô (1984)) A complex-valued function $\hat{F}(\phi)$, $\phi \in S(\mathbb{R}^d)$, is the characteristic functional of an $S'(\mathbb{R}^d)$-valued random variable provided $\hat{F}(\phi)$ satisfies all the three conditions in the preceding theorem.*

The correspondence $X \longmapsto \hat{F}_X$ is injective, i.e., if $\hat{F}_X = \hat{F}_Y$, then $X = Y$ a.s., see Itô (1984).

**Definition 1.4.4** *A stochastic process $\{Y_t, t \geq 0\}$ with values in $S'(\mathbb{R}^d)$, is a collection of random variables $Y_t$ from $(\Omega, \mathcal{F}, P)$ into $(S'(\mathbb{R}^d), \mathcal{B}(S'(\mathbb{R}^d)))$. A stochastic process $\{Y_t, t \geq 0\}$ with values on $S'(\mathbb{R}^d)$ is called Gaussian if all its finite dimensional distributions are Gaussian, i.e., if for every $n \in \mathbb{N}$, all $t_1,\cdots,t_n \in [0,\infty)$ and $\phi_1,\cdots,\phi_n \in S(\mathbb{R}^d)$, the random vector*

$$(\langle \phi_1, Y_{t_1}\rangle, \cdots, \langle \phi_n, Y_{t_n}\rangle)$$

*has a Gaussian distribution on* $\mathbb{R}^n$.

**Theorem 1.4.5** *(Martin-Löf (1976)) Let* $Y := \{Y_t, t \geq 0\}$ *be a centered* $S'(\mathbb{R}^d)$*-valued Gaussian process. Then* $Y$ *is a Markov process if for any fixed* $t > 0$ *the following holds: for all* $t_0 \leq t$ *and* $\varphi \in S(\mathbb{R}^d)$*, there exists* $\hat{\varphi} \in S(\mathbb{R}^d)$ *such that*

$$\mathbb{E}\langle \varphi, Y_s \rangle \langle \psi, Y_t \rangle = \mathbb{E}\langle \hat{\varphi}, Y_{t_0} \rangle \langle \psi, Y_s \rangle, \quad s \leq t_0 \leq t, \tag{1.7}$$

*for all* $s \leq t_0$ *and* $\psi \in S(\mathbb{R}^d)$.

Let $\{T_t, t \geq 0\}$ be a strongly continuous semigroup of continuous linear operators from $S(\mathbb{R}^d)$ into $S(\mathbb{R}^d)$. A linear operator $A : S(\mathbb{R}^d) \longrightarrow S(\mathbb{R}^d)$ is called infinitesimal generator of $\{T_t, t \geq 0\}$ if

$$T_t\psi - \psi = \int_0^t T_s A\psi ds = \int_0^t AT_s\psi ds, \quad t \geq 0, \quad \psi \in S(\mathbb{R}^d).$$

**Theorem 1.4.6** *Let* $Y := \{Y_t, t \geq 0\}$ *be a centered* $S'(\mathbb{R}^d)$*-valued Gaussian process such that, for all* $t \geq 0$,

$$Cov(\langle \varphi, Y_s \rangle, \langle \psi, Y_t \rangle) = Cov(\langle \varphi, Y_s \rangle, \langle T_{t-s}\psi, Y_s \rangle), \quad s \leq t,$$

*where* $\{T_t, t \geq 0\}$ *is a strongly continuous semigroup of linear operators on* $S(\mathbb{R}^d)$ *with infinitesimal generator* $A$*. Then, for all* $\psi \in S(\mathbb{R}^d)$,

$$\langle \psi, Y_t \rangle - \int_0^t \langle A\psi, Y_s \rangle ds, \quad t \geq 0,$$

*is a square-integrable martingale with respect to the filtration* $\mathcal{F}_t = \sigma\{\langle \phi, Y_r \rangle, r \leq t, \phi \in S(\mathbb{R}^d)\}$, $t \geq 0$.

For a proof of this theorem see Fernández (1986).

**Remark 1.4.7** *In our case,* $\{T_t, t \geq 0\}$ *will be the stable semigroup and* $A = \Delta_\alpha$, $\alpha \in (0, 2]$*. For* $\alpha < 2$*, see Theorem 1.3.1.*

Let $F_+$ be the set of all positive locally bounded functions on $[0, \infty)$.

**Theorem 1.4.8** *(Mitoma (1983a)) There exists $p \geq 1$ such that $Y$ is almost surely continuous in the norm $\|\cdot\|_{-p}$ if and only if there exists $g \in F_+$ such that*

$$\sup_{T \in \mathbb{R}_+} \frac{V_T(\phi)}{g(T)} < \infty,$$

*where*

$$V_T(\phi) := \mathbb{E} \sup_{0 \leq t \leq T} \langle \phi, Y_s \rangle^2.$$

## 1.5 Generalized Wiener Process and Langevin Equation

Let $A : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ be a continuous linear operator, and let $Y := \{Y_t, t \geq 0\}$ be a Gaussian process with values in $S'(\mathbb{R}^d)$. Following Bojdecki and Gorostiza (1986), in this section we introduce the concept of generalized Wiener process and give conditions under which $Y$ satisfies a generalized Langevin equation of the form

$$dY_t = A^* Y_t dt + dW_t, \tag{1.8}$$

where $A^*$ is the adjoint operator of $A$ (Treves (1967)), and $W$ is Wiener process with values in $S'(\mathbb{R}^d)$.

**Definition 1.5.1** *A centered Gaussian process $\{W_t, t \geq 0\}$ with values in $S'(\mathbb{R}^d)$ is called* generalized Wiener process *if it has continuous paths and its covariance functional*

$$K(s, \varphi; t, \psi) := \mathbb{E}[\langle \varphi, W_s \rangle \langle \psi, W_t \rangle] \ \ s, t \geq 0, \varphi, \psi \in S(\mathbb{R}^d),$$

*has the form*

$$K(s, \varphi; t, \psi) = \int_0^{t \wedge s} \langle \psi, Q_u \varphi \rangle du,$$

*where the operators $Q_u : S(\mathbb{R}^d) \to S'(\mathbb{R}^d)$ have the following properties:*

1. *For each $u \geq 0$, $Q_u$ is linear, continuous, symmetric and positive.*

2. *For each $\varphi, \psi \in S(\mathbb{R}^d)$, the function $u \longmapsto \langle \psi, Q_u \varphi \rangle$ is continuous from the right and have limits from the left.*

We say that the generalized Wiener process is associated to the family of operators $\{Q_u, u \geq 0\}$.

**Definition 1.5.2** *We say that $Y$ satisfies (in the mild sense) the generalized Langevin equation (1.8) if for each $\phi \in S(\mathbb{R}^d)$,*

$$\langle \phi, Y_t \rangle = \langle \phi, Y_0 \rangle + \int_0^t \langle A\phi, Y_s \rangle ds + \langle \phi, W_t \rangle, \quad t \geq 0. \tag{1.9}$$

**Theorem 1.5.3** *(Bojdecki and Gorostiza (1986)) Assume that the following conditions are true:*

1. *$Y = \{Y_t, t \geq 0\}$ is a continuous centered Gaussian process with values in $S'(\mathbb{R}^d)$ and covariance functional*

$$K(s, \varphi; t, \psi) = Cov(\langle \varphi, Y_s \rangle, \langle \psi, Y_t \rangle), \quad s, t \geq 0, \quad \varphi, \psi \in S(\mathbb{R}^d).$$

2. *For each $\varphi \in S(\mathbb{R}^d)$, the function $s \longmapsto K(s, \varphi; s, \varphi)$ is continuously differentiable.*

3. *$A$ is a continuous operator from $S(\mathbb{R}^d)$ into itself, and generates a strongly continuous semi-group of operators $\{\mathcal{S}_t, t \geq 0\}$ on $S(\mathbb{R}^d)$.*

4. *For any $0 \leq s \leq t$ and $\varphi, \psi \in S(\mathbb{R}^d)$, $K$ satisfies*

$$K(s, \varphi; t, \psi) = K(s, \varphi; s, \mathcal{S}_{t-s}\psi).$$

*Then $Y$ is a Markov process, and there exists a generalized Wiener process $W$ such that $Y$ is solution of (1.8). The Wiener process $W$ is associated to the family of operators $Q = \{Q_u, u \geq 0\}$ defined by*

$$\langle \varphi, Q_u \psi \rangle = \frac{d}{du} K(u, \varphi; u, \psi) - K(u, A\varphi; u, \psi) - K(u, \varphi; u, A\psi), \quad \varphi, \psi \in S(\mathbb{R}^d). \tag{1.10}$$

## 1.6 Spectral measure

**Definition 1.6.1** *A random variable $Y$ with values in $S'(\mathbb{R}^d)$ is called homogeneous, stationary or translation invariant, if for all $\varphi_1, \cdots, \varphi_n \in S(\mathbb{R}^d)$, $n = 1, 2, \cdots,$ and $h \in \mathbb{R}^d$*

$$(\langle \varphi_1, Y \rangle, \cdots, \langle \varphi_n, Y \rangle) \stackrel{d}{=} (\langle \varphi_1(\cdot + h), Y \rangle, \cdots, \langle \varphi_n(\cdot + h), Y \rangle),$$

*where $\stackrel{d}{=}$ means equality in distribution.*

**Theorem 1.6.2** *Let $Y$ be a homogeneous $S'(\mathbb{R}^d)$-valued random variable. Then, its covariance functional can be written in the form*

$$Cov(\langle \varphi, Y \rangle, \langle \psi, Y \rangle) = \int_{\mathbb{R}^d} \hat{\varphi}(x) \overline{\hat{\psi}(x)} \sigma(dz),$$

*where $\hat{\varphi}$ is the Fourier transform of $\varphi$, i.e.,*

$$\hat{\varphi}(z) = \int_{\mathbb{R}^d} e^{ix \cdot y} \varphi(z) dz,$$

*where $x \cdot y$ is the inner product in $\mathbb{R}^d$, and $\sigma$ is a non-negative Radon measure which is called spectral measure.*

## 1.7 Facts from renewal theory

Let $F$ be a distribution function on $\mathbb{R}_+$. The renewal function $U$ associated to $F$ is defined by

$$U(t) = \sum_{n=0}^{\infty} F^{*n}(t),$$

where $F^{*0} \equiv 1$ and $F^{*n}$ denotes the $n$th power convolution of $F$, $n = 1, 2, \ldots$. The renewal function $U$ satisfies the following renewal equation

$$U(t) = 1 + \int_0^t U(t-s) dF(s), \quad t \geq 0.$$

**Remark 1.7.1** *If for* $\lambda > 0$, $1 - F(t) = e^{-\lambda t}$, $t \geq 0$. *Then,* $U(t) = 1 + \lambda t$, $t \geq 0$.

**Lemma 1.7.2** *Assume that* $F$ *has a continuous density* $f$. *Then, the renewal function* $U$ *possesses a continuous density* $u$, *which satisfies*

$$u(t) = f(t) + \int_0^t u(t-s) f(s) ds.$$

**Proof:** See Feller (1968) p. 367.

**Theorem 1.7.3** *(Elementary Renewal Theorem) Assume that* $F$ *is distribution function on the non-negative real line with mean* $0 < \mu < \infty$. *Then,*

$$\lim_{t \longrightarrow \infty} \frac{U(t)}{t} = \frac{1}{\mu}.$$

**Proof:** See Karlin and Taylor (1975) p. 188, or Feller (1968).

**Definition 1.7.4** *A slowly varying function is a positive measurable function* $l$, *defined on a neighborhood* $[A, \infty)$ *of infinity, such that* $l(cx) \sim l(x)$, *i.e.,*

$$\frac{l(cx)}{l(x)} \rightarrow 1, \quad x \rightarrow \infty, \quad \text{for all} \quad c > 0.$$

**Theorem 1.7.5** *Let* $l$ *be a slowly varying function. Then,*

$$1 - F(t) \sim \frac{l(t)}{t^\gamma \Gamma(1-\gamma)},$$

*for some* $0 < \gamma < 1$ *as* $t \rightarrow \infty$ *if, and only if*

$$U(t) \sim \frac{t^\gamma}{l(t) \Gamma(1+\gamma)},$$

*as* $t \rightarrow \infty$.

**Proof:** See Bingham et. al. (1987) p. 361.

## 1.8 Weak convergence

In this section we give a result from Gorostiza and Fernández (1991) which will be used to prove weak convergence of fluctuation processes in the Skorokhod space.

Let $E$ and $F$ be Fréchet nuclear spaces, or strict inductive limits of a sequence of Fréchet nuclear spaces, and let $E'$ and $F'$ be their strong dual, respectively, (see e.g. Treves (1967) for for definitions of these concepts). Let us denote by $D(\mathbb{R}_+, E')$ the space of functions from $\mathbb{R}_+$ to $E'$ which are right-continuous with left limits. We endow $D(\mathbb{R}_+, E')$ with the Skorokhod topology, see Treves (1967), Gelfand and Vilenkin (1966), Mitoma (1981), Mitoma (1983a). The subset of continuous elements of $D(\mathbb{R}_+, E')$ is denoted by $C(\mathbb{R}_+, E')$. Convergence in the Sokorokhod topology is denoted by $\Longrightarrow$, and $\Longrightarrow_f$ means convergence in the sense of finite-dimensional distributions.

**Theorem 1.8.1** *(Gorostiza and Fernández (1991)) Let $\{X_t^n, t \geq 0\}_{n\geq 1}$ be a sequence of processes with paths in $D(\mathbb{R}_+, E')$, and let $X^0$ be a process with paths in $C(\mathbb{R}_+, E')$. Assume that*

*(a) For each $\phi \in E$ there exists $\psi_\phi \in E$ such that for every $n \geq 0$ the process*

$$M_t^n(\phi) := \langle \phi, X_t^n \rangle - \int_0^t \langle \psi_\phi, X_s^n \rangle ds, \quad t \geq 0,$$

*is a martingale.*

*(b) $X^n \Longrightarrow_f X^0$ as $n \to \infty$.*

*(c) For each $T > 0$ and $\phi \in E$ there exists $\eta > 0$ such that*

$$\sup_{n\geq 1} \int_0^T \mathbb{E}|\langle \phi, X_s^n \rangle|^{1+\eta} ds < \infty.$$

*(d) For each $t \geq 0$ and $\phi \in E$ the sequence $\{M_t^n(\phi)\}_{n\geq 1}$ is uniformly integrable.*

*Then $X^n \Longrightarrow X^0$ as $n \to \infty$ in $D(\mathbb{R}_+, E')$.*

**Remark 1.8.2** *It can be proved that conditions (c) and (d) of the above theorem are satisfied if for each $T > 0$ and $\phi \in E$,*

$$\sup_{n\geq 1} \sup_{0\leq t\leq T} \mathbb{E}\langle \phi, X_t^n \rangle^2 < \infty.$$

Let $M := \{M_t, t \geq 0\}$ be a stochastic process with paths on $D([0,\infty), E' \times F')$. Then, $M$ can be seen as follows: $M = (M^1, M^2)$, with $M^1 := \{M^1_t, t \geq 0\}$ and $M^2 := \{M^2_t, t \geq 0\}$ taking values in $D([0,\infty), E')$ and $D([0,\infty), F')$, respectively.

Define $\mathbb{T} : D([0,\infty), E' \times F') \longrightarrow D([0,\infty), E')$ as follows. Given $x := \{(x^1, x^2) = (x^1(t), x^2(t)), t \geq 0\} \in D([0,\infty), E' \times F')$, let

$$\mathbb{T}x = x^1.$$

Then $\mathbb{T}$ is a continuous transformation. An application of the continuous mapping theorem gives the following.

**Lemma 1.8.3** *Let $\{M_n, n = 0, 1, 2, \cdots\}$ be a sequence of stochastic processes on $D([0,\infty), E' \times F')$ and assume that $M_n \Longrightarrow M_0$, as $n \longrightarrow \infty$. Then,*

$$\mathbb{T}M_n \Longrightarrow \mathbb{T}M_0 \equiv M^1_0, \quad \textit{as} \;\; n \longrightarrow \infty. \tag{1.11}$$

## 1.9 List of notations

$\mathcal{B}(E)$: the Borel $\sigma$-field of a topological space $E$.

$\Lambda$: Lebesgue measure on $\mathbb{R}^d$.

$\Longrightarrow$: denotes weak convergence in the Skorokhood space.

$\Longrightarrow_f$: means convergence in the sense of finite-dimensional distributions.

$\delta_x$: Dirac measure at $x$.

$\mathrm{Dom}(\Delta_\alpha)$: domain of $\Delta_\alpha$ on $C_p(\mathbb{R}^d)$.

$\mathrm{Dom}(A)$: domain of the operator $A$.

a.s.: almost sure.

$\sigma\{X_t, t \in \mathbb{I}\}$: $\sigma$-field generated by the random variables $X_t$, $t \in \mathbb{I}$.

$F_+$: the set of all positive locally bounded functions on $[0,\infty)$.

$S(\mathbb{R}^d)$: space of rapidly decreasing functions.

$S'(\mathbb{R}^d)$: strong dual of $S(\mathbb{R}^d)$.

$\{B_t, t \geq 0\}$: spherically symmetric $\alpha$-stable process.

$\langle \phi, \mu \rangle$: denotes $\int \phi d\nu$, where $\phi$ is a measurable function and $\mu$ is a measure.

# Chapter 2

# The Model

We consider a population of individuals or particles living in $\mathbb{R}^d$. Any given particle lives a random amount of time $\tau$ during which it migrates following a spherically symmetric $\alpha$-stable motion, and, at the end of its life, it branches leaving behind a random number $\zeta$ of descendants, all appearing at the parent's death position, and evolving independently under the same rules. We assume that the population starts off from a Poisson population of new particles, with intensity measure $\Lambda$, where $\Lambda$ denotes the Lebesgue measure on $\mathbb{R}^d$.

We assume the classical binary splitting, that is, the probability generating function of $\zeta$ is given by

$$\Phi(s) = \frac{1}{2} + \frac{1}{2}s^2, \quad s \in [-1, 1]. \tag{2.1}$$

Moreover, we suppose that the lifetime $\tau$ has a distribution function $F$ such that $\mathrm{supp}(F) \subset [0, \infty)$, $F(0) = 0$, $F(x) < 1$ for all $x \in [0, \infty)$, and

$$1 - F(u) \sim u^{-\gamma}/\Gamma(1-\gamma) \quad \text{as} \quad u \to \infty \tag{2.2}$$

for some $\gamma \in (0, 1)$, where $\Gamma(\cdot)$ denotes the gamma function. Notice that (2.2) implies that $F$ belongs to the domain of attraction of a $\gamma$-stable law.

We put

$$\Psi(s) = \Phi(1-s) - 1 + s, \quad s \in [-1, 1], \tag{2.3}$$

and write $B \equiv \{B_t, t \geq 0\}$ for the $d$-dimensional $\alpha$-stable process. The $\alpha$-stable transition densities, and the corresponding semigroups of linear operators, are designated, respectively, $\{p_t(x,y),\, t > 0,\, x, y \in \mathbb{R}^d\}$ and $\{\mathcal{S}_t,\, t \geq 0\}$.

Recall that, we write $X_t(A) \equiv X(t,A)$ for the number of individuals living in the Borel set $A \in \mathbb{R}^d$ at time $t \geq 0$; and we notice that the process $X \equiv \{X_t, t \geq 0\}$ takes values in the space of locally finite counting measures on $\mathbb{R}^d$.

Let $Z_t(A)$ be the number of individuals in $A \in \mathcal{B}(\mathbb{R}^d)$ at time $t \geq 0$, in a population starting at time $t = 0$ with a single individual. Given $x \in \mathbb{R}^d$ and $t \geq 0$, we define

$$Q_t\varphi(x) := \mathbb{E}_x\left[1 - e^{-\langle \varphi, Z_t\rangle}\right], \tag{2.4}$$

where $\mathbb{E}_x$ means that the initial particle is located at $x \in \mathbb{R}^d$, and for the moment, let us suppose that $\varphi \in C_c(\mathbb{R}^d)_+$. Note that $Q_t\varphi \equiv 0$, if $\varphi = 0$. Hence, since the initial population is Poissonian, we have that

$$\begin{aligned} \mathbb{E}e^{-\langle \varphi, X_t\rangle} &= \exp\left(-\int \mathbb{E}_x\left[1 - e^{-\langle \varphi, Z_t\rangle}\right] dx\right) \\ &= \exp\left(-\int Q_t\varphi(x)dx\right), \end{aligned} \tag{2.5}$$

where $\mathbb{E}[\cdot]$ denotes expectation starting with a Poisson random field as described above.

Let $\{\tau_k, k \geq 1\}$ be a sequence of i.i.d. random variables with common distribution $F$, and define $\{S_k, k \geq 0\}$ recursively by

$$S_{k+1} = S_k + \tau_k, \quad k \geq 0, \quad S_0 = 0.$$

Define the counting process $\{N_t, t \geq 0\}$ by

$$N_t = \sum_{k=1}^{\infty} 1_{\{S_k \leq t\}}, \quad t \geq 0.$$

Notice that $N_t$ gives the generation number at time $t \geq 0$.

## 2.1 Some moment calculations

Our aim in this section is to calculate the first- and second-order moments of the critical binary branching age-dependent branching particle system. These moments are used latter to prove laws of large numbers and functional central limit theorems for our model. The following two lemmas 2.1.1 and 2.1.2 are borrowed from Kaj and Sagitov (1998). Lemma 2.1.1 gives an integral equation for the function defined in (2.4).

**Lemma 2.1.1** *The function $Q_t\varphi$ solves the equation*

$$Q_t\varphi(x) = \mathbb{E}_x\left[1 - e^{-\varphi(B_t)} - \int_0^t \Psi\left(Q_{t-s}\varphi(B_s)\right)dN_s\right]$$

Given $p = 1, 2, \cdots$, $0 < t_p \le t_{p-1} \le \cdots t_1 < \infty$, $\varphi_1, \varphi_2, \cdots, \varphi_p \in S(\mathbb{R}^d)$ and $\theta_1, \cdots, \theta_p \in \mathbb{R}$ we define $\bar{t} = (t_1, t_2, \cdots, t_p)$, $\bar{t} - s = (t_1 - s, t_2 - s, \cdots, t_p - s)$, $\theta_{(p)} = (\theta_1, \cdots, \theta_p)'$ and

$$Q^p_{\bar{t}}\theta_{(p)}(x) = \mathbb{E}_x\left[1 - e^{-\sum_{j=1}^p \theta_j\langle\varphi_j, Z_{t_j}\rangle}\right].$$

**Lemma 2.1.2** *The function $Q^p_{\bar{t}}\theta_{(p)}$ satisfies*

$$\begin{aligned} Q^p_{\bar{t}}\theta_{(p)}(x) &= \mathbb{E}_x\left[1 - e^{-\sum_{j=1}^p \theta_j\varphi_j(B_{t_j})} - \int_0^{t_1} \Psi(Q^p_{\bar{t}-s}\theta_{(p)}(B_s))dN_s\right. \\ &\quad \left. - \sum_{i=1}^{p-1}\left(1 - e^{-\sum_{j=i+1}^p \theta_j\varphi_j(B_{t_j})}\right)\int_{t_{i+1}}^{t_i} \Psi(Q^i_{\bar{t}-s}\theta_{(i)}(B_s))dN_s\right]. \end{aligned}$$

Using that the initial population is Poissonian we obtain, as in (2.4), that

$$\begin{aligned} \mathbb{E}\left[e^{-\sum_{j=1}^p \theta_j\langle\varphi_j, X_{t_j}\rangle}\right] &= \exp\left(-\int \mathbb{E}_x\left[1 - e^{-\sum_{j=1}^p \theta_j\langle\varphi, Z_{t_j}\rangle}\right]dx\right) \\ &= \exp\left(-\int Q^p_{\bar{t}}\theta_{(p)}(x)dx\right). \end{aligned} \tag{2.6}$$

Given $s, t \geq 0$, $\varphi, \psi \in C_c(\mathbb{R}^d)_+$ and $x \in \mathbb{R}^d$, we define

$$m_x(t,\varphi) := \mathbb{E}_x[\langle \varphi, Z_t\rangle], \tag{2.7}$$

$$C_x(s,\varphi;t,\psi) := \mathbb{E}_x\left[\langle \varphi, Z_s\rangle\langle \psi, Z_t\rangle\right], \tag{2.8}$$

$$m(t,\varphi) := \mathbb{E}\langle \varphi, X_t\rangle,$$

$$C(s,\varphi;t,\psi) := \mathrm{Cov}\left(\langle \varphi, X_s\rangle, \langle \psi, X_t\rangle\right).$$

**Lemma 2.1.3** *For each $t \geq 0$ and $\varphi \in S(\mathbb{R}^d)$,*

$$m(t,\varphi) = \langle \varphi, \Lambda\rangle.$$

**Proof:** First, we note that from (2.6), with $p = 1$,

$$\begin{aligned} m(t,\varphi) &= -\left(\frac{\partial}{\partial\theta}\mathbb{E}e^{-\theta\langle\varphi,X_t\rangle}\right)|_{\theta=0} \\ &= \int \frac{\partial}{\partial\theta}Q_t\theta(x)|_{\theta=0}dx \\ &= \int m_x(t,\varphi)dx. \end{aligned}$$

From Lemma 2.1.1 we see that

$$\frac{\partial}{\partial\theta}Q_t\theta(x) = \mathbb{E}_x\left[\varphi(B_t)e^{-\theta\varphi(B_t)} - \int_0^t \Psi'\left(Q_{t-s}\theta(B_s)\right)dN_s\right],$$

where $Q_t\theta(x)|_{\theta=0} \equiv 0$ and, by criticality, $\Psi'(0) \equiv 0$. Hence,

$$m_x(t,\varphi) = \left(\mathcal{S}_t\varphi\right)(x).$$

Therefore,

$$m(t,\varphi) = \int \left(\mathcal{S}_t\varphi\right)(x)dx = \langle \varphi, \Lambda\rangle,$$

where the last equality follows from the fact that $\Lambda$ is an invariant measure for the $\alpha$-stable semi-group.

**Lemma 2.1.4** *Assume that $0 < s \leq t < \infty$ and $\psi, \varphi \in S(\mathbb{R}^d)$. Then,*

$$C_x(s,\varphi;t,\psi) = \mathbb{E}_x\left[\varphi(B_s)\psi(B_t) + \int_0^s m_{B_r}(t-r,\psi)m_{B_r}(s-r,\varphi)dN_r\right]. \tag{2.9}$$

**Proof:** In order to preserve the notation in Lemma 2.1.2 with $p = 2$, we define $t_1 = t$, $t_2 = s$, $\varphi_1 = \psi$ and $\varphi_2 = \varphi$. Now, we observe that

$$C_x(t_1, \varphi_1; t_2, \varphi_2) = -\frac{\partial^2}{\partial\theta_1\partial\theta_2} Q^2_{\bar{t}}\theta_{(2)}(x)\bigg|_{\theta_1=\theta_2=0^+}, \tag{2.10}$$

where

$$\begin{aligned}\frac{\partial^2}{\partial\theta_1\partial\theta_2} Q^2_{\bar{t}}\theta_{(2)}(x) &= \mathbb{E}_x\Big[-\varphi_1(B_{t_1})\varphi_2(B_{t_2})e^{-\theta_1\varphi(B_{t_1})-\theta_2\varphi_2(B_{t_2})} \\ &\quad -\int_0^{t_2} \Psi''(Q^2_{\bar{t}-r}\theta_{(2)}(B_r))\frac{\partial}{\partial\theta_2}Q^2_{\bar{t}-r}\theta_{(2)}(B_r)\frac{\partial}{\partial\theta_1}Q^2_{\bar{t}-r}\theta_{(2)}(B_r)dN_r \\ &\quad -\int_0^{t_2} \Psi'(Q^2_{\bar{t}-r}\theta_{(2)}(B_r))\frac{\partial^2}{\partial\theta_2\partial\theta_1}Q^2_{\bar{t}-r}\theta_{(2)}(B_r)dN_r \\ &\quad -\varphi_2(B_{t_2})e^{-\theta\varphi_2(B_{t_2})}\int_{t_1}^{t_2} \Psi'(Q^1_{t_2-r}\theta_1(B_r))\frac{\partial}{\partial\theta_1}Q^1_{t_2-r}\theta_1(B_r)dN_r\Big].\end{aligned}$$

Hence, from (2.10) evaluating at $\theta_1 = \theta_2 = 0$ and using that $\Psi'(0) \equiv 0$ and $\Psi''(0) \equiv 1$ we finish the proof.

**Proposition 2.1.5** *Suppose that $0 < s \le t < \infty$, and that $\psi, \varphi \in S(\mathbb{R}^d)$. Then,*

$$C(s, \varphi; t, \psi) = \langle \varphi\mathcal{S}_{t-s}\psi, \Lambda\rangle + \int_0^s \langle (\mathcal{S}_{s-r}\varphi)(\mathcal{S}_{t-r}\psi), \Lambda\rangle dU(r), \tag{2.11}$$

*where $U(r) = \sum_{k=0}^{\infty} F^{*k}(r)$.*

**Proof:** We put $p = 2$ in (2.6) and, as in Lemma 2.1.4, we define $t_1 = t$, $t_2 = s$, $\varphi_1 = \psi$ and $\varphi_2 = \varphi$. Then

$$\mathbb{E}[\langle\varphi_1, X_{t_1}\rangle\langle\varphi_2, X_{t_2}\rangle] = \frac{\partial^2}{\partial\theta_1\partial\theta_2}\exp\left(-\int Q^2_{\bar{t}}\theta_{(2)}(x)dx\right)\bigg|_{\theta_1=\theta_2=0^+}.$$

Hence,

$$\begin{aligned}\mathbb{E}[\langle\varphi_1, X_{t_1}\rangle\langle\varphi_2, X_{t_2}\rangle] &= \Big[-\frac{\partial^2}{\partial\theta_1\partial\theta_2}\int Q^2_{\bar{t}}\theta_{(2)}(x)dx \\ &\quad + \int\frac{\partial}{\partial\theta_1}\int Q^2_{\bar{t}}\theta_{(2)}(x)dx\int\frac{\partial}{\partial\theta_1}\int Q^2_{\bar{t}}\theta_{(2)}(x)dx\Big]\bigg|_{\theta_1=\theta_2=0^+}.\end{aligned}$$

Therefore, we have obtain that

$$\mathbb{E}\left[\langle\varphi_1, X_{t_1}\rangle\langle\varphi_2, X_{t_2}\rangle\right] = \int C_x(t_1,\varphi_1;t_2,\varphi_2)dx + \int m_x(t_1,\varphi_1)dx \int m_x(t_2,\varphi_2)dx,$$

and from Lemma 2.1.4 we get

$$C(s,\varphi;t,\psi) = \int_{\mathbb{R}^d} \mathbb{E}_x\left[\varphi(B_s)\psi(B_t) + \int_0^s m_{B_r}(t-r,\psi)m_{B_r}(s-r,\varphi)dN_r\right]dx. \quad (2.12)$$

**Remark 2.1.6** *Assume that the life times are exponentially distributed with parameter $\lambda > 0$. Then, from Remark 1.7.1 we know that $dU(t) = \lambda dt$. Hence, (2.11) simplifies to*

$$C(s,\varphi;t,\psi) = \langle\varphi\mathcal{S}_{t-s}\psi, \Lambda\rangle + \lambda\int_0^s \langle\left(\mathcal{S}_{s-r}\varphi\right)\left(\mathcal{S}_{t-r}\psi\right), \Lambda\rangle dr,$$

*which is the covariance functional of the branching system with exponential lifetimes.*

# Chapter 3

# Fluctuation limits

In this Chapter we study fluctuation limit theorems under two different rescalings of the critical binary branching particle system defined in Chapter 2 . Namely, *high density* and *space-time* rescaling. The high density rescaling consists in increasing the number of particles by multiplying the initial intensity by a factor $K$, with $K \to \infty$. The work of Martin-Löf (1976) deals with high density limit of a Poissonian system of independent Markovian particles without branching, a physical meaning of this rescaling is also found there. The space-time scaling consists in changing the space-time coordinates $(x, t)$ by $Kx$ and $K^\alpha t$, respectively. These changes are made in order to exploit the self-similarity of the $\alpha$-stable process. See Gorostiza (1983) for the case of exponentially distributed lifetimes and López-Mimbela (1992) for the multi-type case with exponential lifetimes.

In both cases, high density and space-time rescaling, we obtain a strong law of large numbers, and a functional central limit theorem for the fluctuations (around the mean); the latter rendering a generalized ($S'(\mathbb{R}^d)$-valued) limit processes. Also, we investigate continuity, Markov property and generalized Langevin equations for the limit processes.

In order to prove weak convergence to the fluctuation limits in the Skorokhod space, we need to show weak convergence of the finite-dimensional distributions and tightness of the approximating fluctuation processes. Convergence of finite-dimensional distributions is shown using Bochner-Minlos Theorem (by means of convergence of characteristic functions). The proof of tightness in our setting is more difficult than in the classical case of exponentially distributed life times; this is so because $\{X_t, t \geq 0\}$ is not a Markov process, and many of the known results to prove tightness use this property, as well as some martingale properties. We overcome this problem by enlarging the state space in order to get a Markov process, and of course, to obtain martingale properties that would allow us to apply Theorem 1.8.1. To do this, we will conceive the process $X$ as a process imbedded into a bigger space. More precisely, we consider a process $\hat{X} := \{\hat{X}_t := X_t \times \bar{X}_t, t \geq 0\}$, where the process $\hat{X}$ is a Markov process taking values in $D([0,\infty), \mathcal{M}_p(\mathbb{R}^d) \times \mathcal{M}_p(\mathbb{R}_+ \times \mathbb{R}^d))$, for definition of $\mathcal{M}_p(\mathbb{R}_+ \times \mathbb{R}^d)$ see Fleischmann et. al. (2002).

Section 1 is devoted to the study of the high density limit theorem and Section 2 is concerned with the space-time scaling.

## 3.1 High density limit theorem

In this Section we study the so-called *high density limit* for the process $X$ described in the last chapter. Let us denote by $\{X_t^{(K)}, t \geq 0\}$ the branching system with initial intensity $K\Lambda$. Define $\{M_t^{(K)}, t \geq 0\}$ by

$$\langle \varphi, M_t^{(K)} \rangle = \frac{\langle \varphi, X_t^{(K)} \rangle - K\langle \varphi, \Lambda \rangle}{K^{1/2}},$$

for each $t \geq 0$ and $\varphi \in S(\mathbb{R}^d)$. Our purpose is to study the asymptotic behavior of $M^{(K)}$ as $K \longrightarrow \infty$. In fact, we shall prove that $M^{(K)} \Longrightarrow M$ in the Skorokhod space, where $M$ is certain $S'(\mathbb{R}^d)$-valued Gaussian process. Also, we obtain a strong law of large numbers for the process $\{X_t^{(K)}, t \geq 0\}$.

Throughout this Section we assume that $F$ is a general life time distribution function, i.e., a distribution function concentrated in $\mathbb{R}_+$.

Now, we state our main results of this Section. The first concerns a strong law of large numbers for the process $\{X_t^{(K)}, t \geq 0\}$.

**Theorem 3.1.1** *(Law of large numbers) For each $t \geq 0$ and $\varphi \in S(\mathbb{R}^d)$,*

$$\frac{\langle \varphi, X_t^{(K)} \rangle}{K} \longrightarrow \langle \varphi, \Lambda \rangle,$$

*in $L^2(\mathbb{R}^d)$, as $K \longrightarrow \infty$.*

The main result of this section is the following theorem.

**Theorem 3.1.2** *(Functional central limit theorem) As $K \longrightarrow \infty$, $M^{(K)} \Longrightarrow M$ in $D\left(\mathbb{R}_+, S'(\mathbb{R}^d)\right)$, where $M$ is the centered Gaussian $S'(\mathbb{R}^d)$-valued process with covariance functional:*

$$C(s, \varphi; t, \psi) = \langle \varphi \mathcal{S}_{t-s}\psi, \Lambda \rangle + \int_0^s \langle (\mathcal{S}_{s-r}\varphi)(\mathcal{S}_{t-r}\psi), \Lambda \rangle dU(r), \quad 0 \leq s \leq t, \quad \varphi, \phi \in S'(\mathbb{R}^d), \tag{3.1}$$

*where $U(r) = \sum_{k=0}^{\infty} F^{*k}(r)$.*

**Remark 3.1.3** *(a) Notice that theorems 3.1.1 and 3.1.2 are true for any lifetime distribution function, since equation (2.11) is meaningful for any distribution function $F$.*
*(b) From Remark 2.1.6 we observe that if $F$ corresponds to the exponential distribution, then Theorem 3.1.2 renders the result proved in Gorostiza (1983).*

Proofs of these theorems will be given in Section 3.1.2. In the following section we study some properties of the limit process $M$. To prove the weak convergence we will use the continuity property of $M$.

### 3.1.1 Properties of the Limit Process

In this section we shall show some properties for the limit process. Namely, Markov property, a.s. path continuity and generalized Langevin equation.

**Theorem 3.1.4** *(a) The limit process $M$ is a Markov process.*

*(b) For any $\psi \in S(\mathbb{R}^d)$,*

$$\langle \psi, M_t \rangle - \int_0^t \langle \Delta_\alpha \psi, M_s \rangle ds, \quad t \geq 0, \tag{3.2}$$

*is a square integrable martingale with respect to the filtration $\mathcal{F}_t = \sigma\{\langle \phi, M_r \rangle, r \leq t, \phi \in S(\mathbb{R}^d)\}$, $t \geq 0$.*

**Proof:** (a) First, we show that $C(s, \varphi; s, \mathcal{S}_{t-s}\psi) = C(s, \varphi; t, \psi)$ for all $s \leq t$ and $\varphi, \psi \in \mathcal{S}$. In fact,

$$\begin{aligned} C(s, \varphi; s, \mathcal{S}_{t-s}\psi) &= \langle \varphi \mathcal{S}_{s-s} \mathcal{S}_{t-s} \psi, \Lambda \rangle + \int_0^s \langle (\mathcal{S}_{s-r}[(\mathcal{S}_{t-r}\psi)(\mathcal{S}_{s-r}\varphi)], \Lambda \rangle dU(r) \\ &= \langle \varphi \mathcal{S}_{t-s} \psi, \Lambda \rangle + \int_0^s \langle (\mathcal{S}_{t-r}\psi)(\mathcal{S}_{s-r}\varphi), \Lambda \rangle dU(r) \\ &= C(s, \varphi; t, \psi). \end{aligned} \tag{3.3}$$

Hence, the Markov property follows from Theorem 1.4.5. Part (b) follows immediately from (3.3) and Theorem 1.4.6.

**Theorem 3.1.5** *The limit process $M$ has continuous paths almost surely.*

**Proof:** In fact, we will show that there exists $p \geq 1$ such that $M$ is almost surely continuous in the norm $\|\cdot\|_{-p}$. To this end, we will use that that for any given $\phi$,

$$\sup_{T \in \mathbb{R}_+} \frac{V_T(\phi)}{g(T)} < \infty \tag{3.4}$$

for some $g \in F_+$ and

$$V_T(\phi) := \mathbb{E}\left[\sup_{0 \leq t \leq T} \langle \phi, M_t \rangle^2\right].$$

Taking this for granted, the result follows from Theorem 1.4.8. To prove (3.4) we start by observing that

$$\begin{aligned} \langle \phi, M_t \rangle^2 &= \left( \langle \phi, M_t \rangle - \int_0^t \langle \Delta_\alpha \phi, M_s \rangle ds + \int_0^t \langle \Delta_\alpha \phi, M_s \rangle ds \right)^2 \\ &\leq 2 \left( \langle \phi, M_t \rangle - \int_0^t \langle \Delta_\alpha \phi, M_s \rangle ds \right)^2 + 2 \left( \int_0^t \langle \Delta_\alpha \phi, M_s \rangle ds \right)^2, \end{aligned}$$

hence, using Jensen's inequality,

$$\begin{aligned}
\mathbb{E}\sup_{0\le t\le T}\langle\phi,M_t\rangle^2 &\le 2\mathbb{E}\sup_{0\le t\le T}\left(\langle\phi,M_t\rangle-\int_0^t\langle\Delta_\alpha\phi,M_s\rangle ds\right)^2\\
&\quad+2\mathbb{E}\sup_{0\le t\le T}\left(\int_0^t\langle\Delta_\alpha\phi,M_s\rangle ds\right)^2\\
&\le 2\mathbb{E}\sup_{0\le t\le T}\left(\langle\phi,M_t\rangle-\int_0^t\langle\Delta_\alpha\phi,M_s\rangle ds\right)^2\\
&\quad+2T\mathbb{E}\int_0^T\langle\Delta_\alpha\phi,M_s\rangle^2 ds.
\end{aligned}$$

Now, applying Doob's inequality to the martingale (3.2) we have that

$$\begin{aligned}
\mathbb{E}\sup_{0\le t\le T}\left(\langle\phi,M_t\rangle-\int_0^t\langle\Delta_\alpha\phi,M_s\rangle ds\right)^2 &\le 2^2\mathbb{E}\left(\langle\phi,M_T\rangle-\int_0^T\langle\Delta_\alpha\phi,M_s\rangle ds\right)^2\\
&\le 2^3\left[\mathbb{E}\langle\phi,M_T\rangle^2+\mathbb{E}\left(\int_0^T\langle\Delta_\alpha\phi,M_s\rangle ds\right)^2\right]\\
&\le 2^3\left[\mathbb{E}\langle\phi,M_T\rangle^2+T\int_0^T\mathbb{E}\langle\Delta_\alpha\phi,M_s\rangle^2 ds\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}\sup_{0\le t\le T}\langle\phi,M_t\rangle^2 &\le 2^4\left[\mathbb{E}\langle\phi,M_T\rangle^2+T\int_0^T\mathbb{E}\langle\Delta_\alpha\phi,M_s\rangle^2 ds\right]\\
&\quad+2T\int_0^T\mathbb{E}\langle\Delta_\alpha\phi,M_s\rangle^2 ds\\
&= 2^4\mathbb{E}\langle\phi,M_T\rangle^2+(2^4+2)T\int_0^T\mathbb{E}\langle\Delta_\alpha\phi,M_s\rangle^2 ds.
\end{aligned}$$

Using that

$$\mathbb{E}\langle\phi,M_T\rangle^2=C(T,\phi;T,\phi)=\langle\phi^2,\Lambda\rangle+\int_0^T\langle(\mathcal{S}_{T-r}\phi)^2,\Lambda\rangle dU(r)$$

and (1.6), we have that

$$\begin{aligned}
\langle(\mathcal{S}_{t-r}\phi)^2,\Lambda\rangle &= \int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}p_{t-r}(x-y)\phi(y)dy\right]^2 dx\\
&\le \int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}c(t-r)\frac{\phi(y)}{|x-y|^{d+\alpha}}dy\right]^2 dx\\
&= (t-r)^2\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}c\frac{\phi(y)}{|x-y|^{d+\alpha}}dy\right]^2 dx.
\end{aligned}$$

Hence,

$$\mathbb{E}\langle \phi, M_T\rangle^2 \leq \langle \phi^2, \Lambda\rangle + c_\phi \int_0^T (T-r)^2 dU(r),$$

with

$$c_\phi := \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^d} c \frac{\phi(y)}{|x-y|^{d+\alpha}} dy \right]^2 dx.$$

Therefore,

$$\begin{aligned} \mathbb{E} \sup_{0\leq t\leq T} \langle \phi, M_T\rangle^2 &\leq 2^6 \left[ \langle \phi^2, \Lambda\rangle + c_\phi \int_0^T (T-r)^2 dU(r) \right] \\ &\quad + (2^6+2^2)T \int_0^T \left[ \langle \Delta_\alpha \phi^2, \Lambda\rangle + c_{\Delta_\alpha \phi} \int_0^s (s-r)^2 dU(r) \right] ds, \end{aligned}$$

which renders

$$\mathbb{E} \sup_{0\leq t\leq T} \langle \phi, M_T\rangle^2 \leq c_{d,\alpha,\phi} g(T),$$

where $c_{d,\alpha,\phi}$ is a positive constant, and

$$g(t) = t^2 + \int_0^t (t-r)^2 dr + t\int_0^t \int_0^s (s-r)^2 dU(r) ds,$$

so that $g$ is a positive locally bounded function on $[0,\infty)$. This completes the proof. □

**Theorem 3.1.6** *(Langevin equation) Assume that $F$ has a continuous density $f$. Then, the process $M$ satisfies the generalized Langevin equation*

$$\begin{aligned} dM_t &= \Delta_\alpha^* M_t + d\mathcal{W}_t, \qquad (3.5) \\ M_0 &= W, \end{aligned}$$

*where $W$ is a centered spatial white noise and $\mathcal{W}$ is the Wiener process associated to the family of operators $\{Q_t, t\geq 0\}$ such that for each $\varphi, \psi \in S(\mathbb{R}^d)$,*

$$\langle \varphi, Q_t \psi\rangle = \langle \varphi\psi, \Lambda\rangle u(t) - 2\langle \varphi \Delta_\alpha \psi, \Lambda\rangle, \qquad (3.6)$$

*where $u(t) = dU(t)/dt$.*

**Proof:** We will show that $M$ satisfies all the conditions of Theorem 1.5.3. Condition 1 follows from Theorem 1.4.8, Condition 3 holds by hypothesis and Condition 4 follows from Theorem 1.4.5. It remains to show Condition 2. We have that, for each $t \geq 0$ and $\varphi \in S(\mathbb{R}^d)$,

$$C(t,\varphi;t,\varphi) = \langle \varphi^2, \Lambda \rangle + \int_0^t \langle (\mathcal{S}_{t-r}\varphi)^2, \Lambda \rangle dU(r),$$

and, because of Lemma 1.7.2,

$$C(t,\varphi;t,\varphi) = \langle \varphi^2, \Lambda \rangle + \int_0^t \langle (\mathcal{S}_{t-r}\varphi)^2, \Lambda \rangle u(r) dr.$$

Hence, the function $t \longmapsto C(t,\varphi;t,\varphi)$ is continuously differentiable. Then, applying Theorem 1.5.3 we get the Langevin equation. It remains to show equation (3.6). Notice that for $s = t$, (3.1) can be written as follows

$$C(t,\varphi;t,\psi) = \langle \varphi\psi, \Lambda \rangle + \int_0^t \langle \varphi(\mathcal{S}_{2(t-r)}\psi), \Lambda \rangle u(r) dr, \quad 0 \leq t, \quad \varphi, \phi \in S'(\mathbb{R}^d). \tag{3.7}$$

Therefore,

$$\frac{d}{dt} C(t,\varphi;t,\psi) = \langle \varphi\psi, \Lambda \rangle u(t) + 2 \int_0^t \langle \varphi(\mathcal{S}_{2(t-r)}\Delta_\alpha \psi), \Lambda \rangle u(r) dr.$$

Hence,

$$\begin{aligned} \langle \varphi, Q_t \varphi \rangle &= \frac{d}{dt} C(t,\varphi;t,\varphi) - 2C(t,\Delta_\alpha\varphi;t,\varphi) \\ &= \langle \varphi\psi, \Lambda \rangle u(t) + 2 \int_0^t \langle \varphi(\mathcal{S}_{2(t-r)}\Delta_\alpha \psi), \Lambda \rangle u(r) dr \\ &\quad -2\langle (\Delta_\alpha\varphi)\varphi, \Lambda \rangle u(t) - 2 \int_0^t \langle \varphi(\mathcal{S}_{2(t-r)}\Delta_\alpha \psi), \Lambda \rangle u(r) dr \\ &= \left( \langle \varphi^2, \Lambda \rangle - 2\langle (\Delta_\alpha\varphi)\varphi, \Lambda \rangle \right) u(t). \end{aligned}$$

Then, equation (3.6) can be deduced from

$$\langle \varphi, Q_t \psi \rangle = \frac{1}{2} \left[ \langle (\varphi + \psi), Q_t(\varphi + \psi) \rangle - \langle \varphi, Q_t \varphi \rangle - \langle \psi, Q_t \psi \rangle \right].$$

**Remark 3.1.7** *(a) The assumption in the theorem above that $F$ has a continuous density cannot be dropped; without such assumption we cannot guarantee differentiability of the* *function* $t \longmapsto C(t,\varphi;t,\varphi)$.

*(b) Assuming that* $F(t) = 1 - e^{-\lambda t}$, $t \geq 0$, *and* $\alpha = 2$ *we get that* $u(t) \equiv \lambda$, *see Remark 1.7.1. Hence, (3.6) is equivalent to*

$$\langle \varphi, Q_t \psi \rangle = \lambda \langle \varphi\psi, \Lambda \rangle + \langle \nabla\varphi \cdot \nabla\psi, \Lambda \rangle,$$

*which recovers a result from Gorostiza (1983) for critical binary branching.*

*(c) By Remark (a) in Theorem 3.6 from Bojdecki and Gorostiza (1986), without any regularity condition on F we still have*

$$\langle \varphi, M_t \rangle = \langle \varphi, W \rangle + \int_0^t \langle \varphi, M_s \rangle ds + \langle \varphi, \mathcal{W}_t \rangle, \quad t \geq 0,$$

*where* $\{\mathcal{W}_t, t \geq 0\}$ *is a continuous* $S'(\mathbb{R}^d)$*-valued Gaussian process with covariance functional*

$$\begin{aligned} \mathbb{E}\left[\langle \varphi, W_s \rangle \langle \varphi, W_t \rangle\right] &= \mathcal{K}(s \wedge t, \varphi; s \wedge t, \psi) \\ &\quad - \int_0^{s \wedge t} (\mathcal{K}(u, \Delta_\alpha \varphi; u, \psi) - \mathcal{K}(u, \varphi; u, \Delta_\alpha \psi)) du, \end{aligned}$$

*for all* $s, t \geq 0$ *and* $\varphi, \psi \in S(\mathbb{R}^d)$.

*(d) In case that* $\alpha = 2$, *equation (3.5) has the meaning given in Section 5 Chapter 1. When* $\alpha < 2$, *(3.5) has to be understood in a generalized sense, because of* $\Delta_\alpha S(\mathbb{R}^d) \not\subseteq S(\mathbb{R}^d)$, *see Dawson and Gorostiza (1990).*

**Theorem 3.1.8** *(Spectral Measure) For any* $t \geq 0$, $M_t$ *is a homogeneous* $S'(\mathbb{R}^d)$*-valued random field whose spectral measure has a density* $\sigma_t(z)$ *given by*

$$\sigma_t(z) = (2\pi)^{-d} + \int_0^t e^{-2(t-r)|z|^\alpha} dU(r), \quad z \in \mathbb{R}^d.$$

**Proof:** From (3.1) we have that, for each $t \geq 0$ and $\varphi, \psi \in S(\mathbb{R}^d)$,

$$\mathrm{Cov}\left(\langle \varphi, M_t \rangle, \langle \psi, M_t \rangle\right) = \langle \varphi\psi, \Lambda \rangle + \int_0^t \langle (\mathcal{S}_{t-r}\varphi)(\mathcal{S}_{t-r}\psi), \Lambda \rangle dU(r).$$

Hence, by Plancherel's formula

$$\langle \varphi\psi, \Lambda \rangle = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{\varphi}(x) \bar{\hat{\psi}}(x) dx$$

and

$$\widehat{\mathcal{S}_t\varphi}(x) = e^{-t|x|^\alpha}\bar{\hat{\varphi}}(x),$$

we get that

$$\int_0^t \langle(\mathcal{S}_{t-r}\varphi)(\mathcal{S}_{t-r}\psi), \Lambda\rangle dU(r) = \int_{\mathbb{R}^d} \hat{\varphi}(x)\bar{\hat{\psi}}(x)\int_0^t e^{-2(t-r)|x|^\alpha}dU(r)\, dx.$$

Therefore,

$$\text{Cov}\left(\langle\varphi, M_t\rangle, \langle\psi, M_t\rangle\right) = \int_{\mathbb{R}^d} \hat{\varphi}(x)\bar{\hat{\psi}}(x)\sigma_t(x)dx,$$

with $\sigma_t$ as in the theorem.

### 3.1.2 Proofs of the high density limit theorems

Here we give the proofs of the limits theorems announced at the beginning of the present section.

Notice that, from Proposition 2.1.5, for $0 \le t_1 \le t_2 < \infty$ and $\varphi_1, \varphi_2 \in S(\mathbb{R}^d)$,

$$\text{Cov}(\langle\varphi_1, X_{t_1}^{(K)}\rangle, \langle\varphi_2, X_{t_2}^{(K)}\rangle) = KC(t_1, \varphi_1; t_2, \varphi_2). \tag{3.8}$$

The next lemma gives convergence of finite-dimensional distributions of $M^{(K)}$ towards $M$.

**Lemma 3.1.9** *$M^{(K)} \Longrightarrow_f M$ as $K \longrightarrow \infty$, i.e., for each $p \ge 1$, $0 < t_p \le t_{p-1} \le \cdots \le t_1 < \infty$, $\varphi_1, \cdots, \varphi_p \in S(\mathbb{R}^d)$ and $\theta_1, \cdots, \theta_p \in \mathbb{R}$,*

$$\mathbb{E}\left[e^{i\sum_{j=1}^p \theta_j\langle\varphi_j, M_{t_j}^{(K)}\rangle}\right] \longrightarrow \exp\left(-\frac{1}{2}\sum_{j=1}^p\sum_{k=1}^p \theta_j\theta_k C(t_j, \varphi_j; t_k, \varphi_k)\right),$$

*as $K \to \infty$.*

**Proof:** The proof of this result relies on Bochner-Minlos-Sasonv's theorem (Itô (1984)), which is the counterpart of Lévy's Continuity Theorem in nuclear spaces. First we note that

$$
\begin{aligned}
\mathbb{E}\left[e^{i\sum_{j=1}^{p}\theta_j\langle\varphi_j,M_{t_j}^{(K)}\rangle}\right] &= \mathbb{E}\left[\exp\left(i\sum_{j=1}^{p}\theta_j\frac{\langle\varphi_j,X_{t_j}^{(K)}\rangle - K\langle\varphi_j,\Lambda\rangle}{K^{1/2}}\right)\right] \\
&= \mathbb{E}\left[\exp\left(+i\sum_{j=1}^{p}\theta_j K^{-1/2}\langle\varphi_j,X_{t_j}^{(K)}\rangle\right)\right] \\
&\quad \times\exp\left(-iK^{1/2}\sum_{j=1}^{p}\theta_j\langle\varphi_j,\Lambda\rangle\right) \\
&= \exp\left(-K\int_{\mathbb{R}^d}\mathbb{E}_x\left[1-e^{i\sum_{j=1}^{p}\theta_j K^{-1/2}\langle\varphi_j,Z_{t_j}\rangle}\right]dx\right) \\
&\quad \times\exp\left(-iK^{1/2}\sum_{j=1}^{p}\theta_j\langle\varphi_j,\Lambda\rangle\right) \\
&= \exp\left(-\frac{1}{2}\int_{\mathbb{R}^d}\mathbb{E}_x\left(\sum_{j=1}^{p}\theta_j\langle\varphi_j,Z_{t_j}\rangle\right)^2 dx\right) \\
&\quad \times\exp\left(\int K\left[\mathbb{E}_x e^{i\sum_{j=1}^{p}K^{-1/2}\theta_j\langle\varphi_j,Z_{t_j}\rangle} - 1\right.\right. \\
&\quad \left.\left. -iK^{-1/2}\sum_{j=1}^{p}\theta_j\mathbb{E}_x\langle\varphi_j,Z_{t_j}\rangle + \frac{1}{2}K^{-1}E_x\left(\sum_{j=1}^{p}\theta_j\langle\varphi_j,Z_{t_j}\rangle\right)^2\right]dx\right),
\end{aligned}
$$

where the integrand converges to 0, as $K \longrightarrow \infty$, and is bounded by $c\sum_{j=1}^{p}\theta_j^2\mathbb{E}_x\langle\varphi_j,Z_{t_j}\rangle^2$ for some constant $c > 0$ (see Breiman (1992) Proposition 8.44). Hence,

$$
\lim_{K\to\infty}\mathbb{E}\left[e^{i\sum_{j=1}^{p}\theta_j\langle\varphi_j,M_{t_j}^{(K)}\rangle}\right] = \exp\left(-\frac{1}{2}\int_{\mathbb{R}^d}\mathbb{E}_x\left(\sum_{j=1}^{p}\theta_j\langle\varphi_j,Z_{t_j}\rangle\right)^2 dx\right).
$$

Then, we finish the proof by observing that

$$
\int_{\mathbb{R}^d}\mathbb{E}_x\left(\sum_{j=1}^{p}\theta_j\langle\varphi_j,Z_{t_j}\rangle\right)^2 dx = \sum_{j=1}^{p}\sum_{k=1}^{p}\theta_j\theta_k C(t_j,\varphi_j;t_k,\varphi_k).
$$

□

**Proof of Theorem 3.1.1.** We need to show that, as $K \longrightarrow \infty$,

$$\mathbb{E}\left(\frac{\langle \varphi, X_t^{(K)}\rangle}{K} - \langle \varphi, \Lambda\rangle\right)^2 \longrightarrow 0,$$

for all $\varphi \in S(\mathbb{R}^d)$. Note that, from (3.8),

$$\begin{aligned}
\mathbb{E}\left(\frac{\langle \varphi, X_t^{(K)}\rangle}{K} - \langle \varphi, \Lambda\rangle\right)^2 &= \frac{1}{K^2}\mathbb{E}\left(\langle \varphi, X_t^{(K)}\rangle - K\langle \varphi, \Lambda\rangle\right)^2 \\
&= \frac{1}{K^2}\mathrm{Var}\left(\langle \varphi, X_t^{(K)}\rangle\right) \\
&= \frac{1}{K}C(t,\varphi;t,\varphi).
\end{aligned}$$

Letting $K \longrightarrow \infty$ yields the result. □

We conclude this Section by giving the proof of Theorem 3.1.2. We recall that, by a well known result of Mitoma (1983b), to show tightness of the sequence $\{M_t^{(K)}, t \geq 0\}$, $K = 1, 2, \cdots$, is enough to prove tightness of the sequence of the real-valued process $\{\langle \varphi, M_t^{(K)}\rangle, t \geq 0\}$, $K = 1, 2, \cdots$, for each $\varphi \in S'(\mathbb{R}^d)$. The following lines uses some notation from the Appendix.

Consider the process $\hat{X} := \{X_t \times \bar{X}_t, t \geq 0\}$, where $\{X_t, t \geq 0\}$ is the branching system defined in Chapter 2, and $\{\bar{X}_t, t \geq 0\}$ is the Markovianized branching system introduced in Appendix A. The process $\hat{X}$ is a Markov process with paths in the Skorokhod space $D([0,\infty), \mathcal{M}(\mathbb{R}^d) \times \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}^d))$.

We next give the infinitesimal generator of the process $\hat{X}$ for certain cylindrical functions. Define

$$g(\mu_1, \mu_2) := G(\langle \varphi, \mu_1\rangle) \text{ for } \varphi \in S(\mathbb{R}^d),\ \mu_1 \in \mathcal{M}(\mathbb{R}^d),\ \mu_2 \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}^d),$$

where $G \in C^3(\mathbb{R})$ is such that $G''' \equiv 0$. It can be seen that the infinitesimal generator

$\mathcal{G}$, is given by

$$\begin{aligned}\mathcal{G}g(\mu_1,\mu_2) &= \langle \Delta_\alpha\varphi,\mu_1\rangle G'(\langle\varphi,\mu_1\rangle) + \frac{1}{2}\langle \Delta_\alpha\varphi^2 - 2\varphi\Delta_\alpha\varphi,\mu_1\rangle G''(\langle\varphi,\mu_1\rangle) \\ &\quad + \langle \lambda(*), \sum_{k=0}^{\infty} p_k[G(\langle\varphi,\mu_1+(k-1)\delta_\cdot\rangle) - G(\langle\varphi,\mu_1\rangle)],\mu_2\rangle, \qquad (3.9)\end{aligned}$$

where $\lambda(u) = f(u)/(1-F(u))$, $u \geq 0$ and for the notation $\langle\psi(*,\cdot),\mu_2\rangle$ see comment following (A.4); this can be done by expanding $G$ in Taylor's series and then proceeding as in Section A.0.2 in the Appendix A.

Putting $G(y) = y$ for all $y \in \mathbb{R}$, from (3.9) we get that

$$\begin{aligned}\mathcal{G}g(\mu_1,\mu_2) &= \langle \Delta_\alpha\varphi,\mu_1\rangle + \langle\lambda(*), \sum_{k=0}^{\infty} p_k(k-1)\varphi(\cdot),\mu_2\rangle \\ &= \langle \Delta_\alpha\varphi,\mu_1\rangle,\end{aligned}$$

where the second equality follows from criticality of the branching. Then, from the Markov property we have that

$$Y_t(\varphi) := \langle\varphi, X_t\rangle - \int_0^t \langle\Delta_\alpha\varphi, X_s\rangle ds, \;\; t \geq 0 \;\text{ and }\; \varphi \in S(\mathbb{R}^d), \qquad (3.10)$$

is a martingale (with respect to the filtration generated by the process $\hat{X}$).

**Proof of Theorem 3.1.2.** We will show that the sequence $\{M^{(K)}, K = 1,2,\cdots\}$ satisfies all the conditions in Theorem 1.8.1. First we note that, by Theorem 3.1.5, the process $M$ possesses continuous paths. Condition (b) is proved in Lemma 3.1.9. To prove conditions (c) and (d) we show that $M^{(K)}$ satisfies Remark 1.8.2. In fact, from (3.8) we can see that

$$\mathbb{E}\langle\varphi, M_t^{(K)}\rangle^2 = \langle\varphi^2,\Lambda\rangle + \int_0^t \langle(\mathcal{S}_{t-r}\varphi)^2,\Lambda\rangle dU(r), \qquad (3.11)$$

for each $t \geq 0$ and $\varphi \in S(\mathbb{R}^d)$. Note that, (3.11) can be bounded from above as follows

$$\mathbb{E}\langle\varphi, M_t^{(K)}\rangle^2 \leq \langle\varphi^2,\Lambda\rangle + \int_0^t \langle(\mathcal{S}_{t-r}|\varphi|)^2,\Lambda\rangle dU(r).$$

Hence, without loss of generality we can assume that $\varphi > 0$. Now, we observe that

$$\begin{aligned}\langle(\mathcal{S}_{t-r}\varphi)^2, \Lambda\rangle &= \langle\frac{\mathcal{S}_{t-r}\varphi}{\phi_p}\phi_p\mathcal{S}_{t-r}\varphi, \Lambda\rangle \\ &\leq |\varphi|_p\langle\phi_p\mathcal{S}_{t-r}\varphi, \Lambda\rangle \\ &\leq |\varphi|_p\langle\mathcal{S}_{t-r}\varphi, \Lambda\rangle \\ &= |\varphi|_p\langle\varphi, \Lambda\rangle. \qquad (3.12)\end{aligned}$$

Hence

$$\begin{aligned}\sup_{0\leq t\leq T}\mathbb{E}\langle\varphi, M^{(K)}\rangle^2 &\leq \sup_{0\leq t\leq T}\left(\langle\varphi^2, \Lambda\rangle + \int_0^t |\varphi|_p\langle\varphi, \Lambda\rangle dU(r)\right) \\ &\leq \langle\varphi^2, \Lambda\rangle + |\varphi|_p\langle\varphi, \Lambda\rangle U(T),\end{aligned}$$

and therefore,

$$\sup_{K\geq 1}\sup_{0\leq t\leq T}\mathbb{E}\langle\varphi, M^{(K)}\rangle^2 < \infty.$$

It remains to verify Condition (a), and for this we use the Markovianized process discussed above. From (3.10) we know that for all $\varphi \in S(\mathbb{R}^d)$,

$$\langle\varphi, X_t\rangle - \int_0^t \langle\Delta_\alpha\varphi, X_s\rangle ds, \quad t \geq 0,$$

is a martingale. Therefore, for all $\varphi \in S(\mathbb{R}^d)$,

$$\langle\varphi, M_t^{(K)}\rangle - \int_0^t \langle\Delta_\alpha\varphi, M_s^{(K)}\rangle ds, \quad t \geq 0, \qquad (3.13)$$

is also a martingale since $\langle\Delta_\alpha\varphi, \Lambda\rangle = 0$, seen as a process in $D\left([0,\infty), S'(\mathbb{R}^d) \times S'(\mathbb{R}^{d+1})\right)$.

Notice that, Lemma 3.1.9 gives $M^{(K)} \Longrightarrow_f M$, from this follows that $(M^{(K)}, 0) \Longrightarrow_f (M, 0)$. Hence, we have shown that $(M^{(K)}, 0) \Longrightarrow (M, 0)$, this convergence holds in the space $D\left([0,\infty), S'(\mathbb{R}^d) \times S'(\mathbb{R}^{d+1})\right)$. Finally, by Lemma 1.8.3 we get that, $M^{(K)} \Longrightarrow M$ as $K \longrightarrow \infty$, in $D\left([0,\infty), S'(\mathbb{R}^d)\right)$. This completes the proof. □

## 3.2 Space-time scaling

This Section is dedicated to the study of the so-called *space-time scaling* limit, in which the coordinates in space-time are $Kx$ and $K^\alpha t$, respectively, and $K \to \infty$. Let $\{X_t^{2,K}, t \geq 0\}$ denote the resulting branching system. Then, for each $K = 1, 2, \ldots$,

$$\langle \varphi, X_t^{2,K} \rangle = \langle \varphi^K, X_{k^\alpha t} \rangle, \;\; t \geq 0, \;\; \varphi \in S(\mathbb{R}^d), \tag{3.14}$$

we recall that $\varphi^K(x) = \varphi(x/K)$, $x \in \mathbb{R}^d$. The fluctuation process $\{M_t^{2,K}, t \geq 0\}$ is defined by

$$\langle \varphi, M_t^{2,K} \rangle = \frac{\langle \varphi^K, X_{K^\alpha t} \rangle - \mathbb{E}\langle \varphi^K, X_{K^\alpha t} \rangle}{K^{(d+\alpha\gamma)/2}}, \quad t \geq 0, \;\; \varphi \in S(\mathbb{R}^d). \tag{3.15}$$

Throughout this Chapter we assume that $d > \alpha\gamma$, which according to Vatutin and Wakolbinger (1999) and Fleischmann et. al. (2002), corresponds to supercritical dimension for persistence, i.e., in this case the process is persistent. Notice that persistence of our branching particle system holds also at the critical dimension $d = \alpha\gamma$, see Vatutin and Wakolbinger (1999) and Fleischmann et. al. (2002). As in many other cases, the critical dimension is much more difficult to handle and requires a more delicate treatment. If we do not restrict to $d > \alpha\gamma$ we can not prove that the error term (involving third order moments) in the Taylor expansion goes to zero, see Lemma 3.2.10 below. Thus we can not end up with a Gaussian process. Hence, it would be interesting to investigate what happens at the critical dimension. For example, looking for another normalizing constant we can try to show some kind of convergence.

We give the statements of the limit theorems of this section. The first one is a strong law of large numbers of the process $\{M_t^{2,K}, t \geq 0\}$.

**Theorem 3.2.1** *(Law of large numbers) Assume that $d > \alpha\gamma$. Then, for each $t \geq 0$*

$$\frac{\langle \varphi^K, X_{K^\alpha t} \rangle}{K^d} \longrightarrow \langle \varphi, \Lambda \rangle, \;\; \varphi \in S(\mathbb{R}^d),$$

*in $L^2(\mathbb{R}^d)$, as $K \longrightarrow \infty$.*

The main result of this section is the following theorem. From now on, the notation $d(u^\gamma)$ should be understood as $\gamma u^{\gamma-1}du$.

**Theorem 3.2.2** *(Functional central limit theorem) Assume that $d > \alpha\gamma$. Then, $M^{2,K} \Longrightarrow M^{(2)}$ as $K \longrightarrow \infty$, in the Skorokhod space $D(\mathbb{R}_+, S'(\mathbb{R}^d))$, where $M^{(2)}$ is a centered Gaussian $S'(\mathbb{R}^d)$-valued process with covariance functional*

$$\mathcal{K}(t_1,\varphi_1;t_2,\varphi_2) := \frac{1}{\Gamma(1+\gamma)}\int_0^{t_2}\int_{\mathbb{R}^d}(\mathcal{S}_{t_2-u}\varphi_2)(x)(\mathcal{S}_{t_1-u}\varphi_1)(x)dx\,d(u^\gamma), \tag{3.16}$$

*for all $0 \le t_2 \le t_1 < \infty$ and $\varphi_1, \varphi_2 \in S(\mathbb{R}^d)$.*

**Remark 3.2.3** *Using the elementary renewal theorem 1.7.3 we can prove that theorems 3.2.1 and 3.2.2 still hold for a general lifetime distribution function with finite mean $\mu > 0$. The corresponding statements of these theorems remain the same as above but with $\gamma = 1$, and the covariance functional in Theorem 3.2.2 changes to*

$$\mathcal{K}(t_1,\varphi_1;t_2,\varphi_2) := \frac{1}{\mu}\int_0^{t_2}\int_{\mathbb{R}^d}(\mathcal{S}_{t_2-u}\varphi_2)(x)(\mathcal{S}_{t_1-u}\varphi_1)(x)dx\,du.$$

*In this way, we recover the classical known result for exponentially distributed lifetimes with rate $1/\mu$.*

### 3.2.1 Properties of the limit process

In this Section, we discuss continuity, Markov property and generalized Langevin equation of the fluctuation limit process.

**Theorem 3.2.4** *(a) $M^{(2)}$ is a Markov process.*
*(b) For any $\phi \in S(\mathbb{R}^d)$,*

$$\langle \phi, M_t^{(2)}\rangle - \int_0^t \langle \Delta_\alpha \phi, M_r^{(2)}\rangle dr, \ \ t \ge 0,$$

*is a square-integrable martingale with respect to the filtration $\mathcal{F}_t = \sigma\{\langle\phi, M_s^{(2)}\rangle, 0 \le s \le t, \phi \in S(\mathbb{R}^d)\}$, for each $t \ge 0$.*

**Proof:** (a) Given $0 \leq t_2 \leq t_1 < \infty$ and $\varphi_1, \varphi_2 \in S(\mathbb{R}^d)$, we have by the semi-group property of $\mathcal{S}$,

$$\begin{aligned}
\mathcal{K}(t_2, \varphi_2; t_2, \mathcal{S}_{t_1-t_2}\varphi_1) &= \frac{1}{\Gamma(1+\gamma)} \int_0^{t_1 \wedge t_2} \langle (\mathcal{S}_{t_2-r}(\mathcal{S}_{t_1-t_2}\varphi_1)) (\mathcal{S}_{t_2-r}\varphi_2), \Lambda \rangle d(r^\gamma) \\
&= \frac{1}{\Gamma(1+\gamma)} \int_0^{t_1 \wedge t_2} \langle (\mathcal{S}_{t_1-r}\varphi_1) (\mathcal{S}_{t_2-r}\varphi_2), \Lambda \rangle d(r^\gamma) \\
&= \mathcal{K}(t_2, \varphi_2; t_1, \varphi_1). \qquad (3.17)
\end{aligned}$$

To finish the proof we apply Theorem 1.4.5 with $\hat{\varphi} = S_{t_1-t_2}\varphi$.

The second part follows from (3.17), as in (b) of Theorem 3.1.4. □

**Theorem 3.2.5** *There exists $p \geq 1$ such that the process $M^{(2)}$ has a continuous version in the norm $\|\cdot\|_{-p}$.*

**Proof:** In the same way as in the proof of Theorem 1.4.8, we can see that

$$\mathbb{E} \sup_{0 \leq t \leq T} \langle \varphi, M_t^{(2)} \rangle^2 \leq 2^6 \mathbb{E} \langle \varphi, M_T^{(2)} \rangle^2 + (2^6 + 2^2) T \int_0^T \mathbb{E} \langle \Delta_\alpha \varphi, M_s^{(2)} \rangle^2 ds.$$

But, from Theorem 3.2.2 we know that,

$$\begin{aligned}
\mathbb{E} \langle \varphi, M_t^{(2)} \rangle^2 &= \mathcal{K}(t, \varphi; t, \varphi) \\
&= \frac{1}{\Gamma(1+\gamma)} \int_0^t \langle (\mathcal{S}_{t-r}\varphi)^2, \Lambda \rangle d(r^\gamma),
\end{aligned}$$

and, as in the proof of Theorem 3.1.5,

$$\langle (\mathcal{S}_{t-r}\varphi)^2, \Lambda \rangle \leq c_\varphi (t-r)^2.$$

Hence,

$$\begin{aligned}
\mathbb{E} \langle \varphi, M_t^{(2)} \rangle^2 &\leq \frac{c_\varphi}{\Gamma(1+\gamma)} \int_0^t (t-r)^2 \gamma r^{\gamma-1} dr \\
&= \frac{\gamma c_\varphi}{\Gamma(1+\gamma)} \int_0^1 (1-u)^2 u^{\gamma-1} du t^\gamma \\
&= \frac{\gamma c_\varphi}{\Gamma(1+\gamma)} \frac{\Gamma(\gamma+3)}{\Gamma(\gamma)\Gamma(3)} t^\gamma,
\end{aligned}$$

where in the first equality we made the change of variables $u = r/t$, and the second identity comes from the form of the Beta distribution. Then,

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}\langle\varphi, M_t^{(2)}\rangle^2 &\le 2^6\frac{\gamma c_\varphi}{\Gamma(1+\gamma)}\frac{\Gamma(\gamma+3)}{\Gamma(\gamma)\Gamma(3)}T^\gamma + (2^6+2^2)T\int_0^T \mathrm{const}(\varphi)s^\gamma ds \\ &= 2^6 c_\varphi T^\gamma + (2^6+2^2)\frac{\gamma\mathrm{const}(\varphi)}{\Gamma(1+\gamma)}\frac{\Gamma(\gamma+3)}{\Gamma(\gamma)\Gamma(3)}T\frac{T^{\gamma+1}}{\gamma+1},\end{aligned}$$

and therefore,

$$\mathbb{E}\sup_{0\le t\le T}\langle\varphi, M_t^{(2)}\rangle^2 \le c^2_{d,\gamma,\varphi,\alpha}g(T),$$

where $c^2_{d,\gamma,\varphi,\alpha} > 0$ is a constant, and $g(t) = t^\gamma + t^{2+\gamma}$ for all $t \ge 0$. Noticing that $g \in F_+$, we get that

$$\sup_{T\ge 0}\frac{\mathbb{E}\sup_{0\le t\le T}\langle\varphi, M_t^{(2)}\rangle^2}{g(T)} = c^2_{d,\gamma,\varphi,\alpha} < \infty.$$

An application of Theorem 1.4.8 finishes the proof. □

**Theorem 3.2.6** *(Langevin equation) For each $\phi \in S(\mathbb{R}^d)$,*

$$\langle\phi, M_t^{(2)}\rangle = \langle\phi, M_0^{(2)}\rangle + \int_0^t \langle\Delta_\alpha\phi, M_s^{(2)}\rangle ds + \langle\phi, W_t\rangle, \quad t \ge 0,$$

*where $\{W_t, t \ge 0\}$ is a continuous $S'(\mathbb{R}^d)$-valued Gaussian process with covariance functional*

$$\begin{aligned}\mathbb{E}\left[\langle\varphi, W_s\rangle\langle\varphi, W_t\rangle\right] &= \mathcal{K}(s\wedge t,\varphi; s\wedge t,\psi) \\ &\quad - \int_0^{s\wedge t}(\mathcal{K}(u,\Delta_\alpha\varphi; u,\psi) - \mathcal{K}(u,\varphi; u,\Delta_\alpha\psi))du, \\ &\quad s,t\ge 0, \quad \varphi,\psi\in S(\mathbb{R}^d).\end{aligned}$$

**Proof:** This follows directly from Remark (a) in Theorem 3.6 of Bojdecki and Gorostiza (1986). □

**Theorem 3.2.7** *(Spectral Measure) For any $t \geq 0$, $M_t^{(2)}$ is a homogeneous $S'(\mathbb{R}^d)$-valued random field whose spectral measure, $\sigma_t(z)$, is given by*

$$\sigma_t(z) = \int_0^t e^{-2(t-r)|z|^\alpha} d(r^\gamma), \quad z \in \mathbb{R}^d.$$

**Proof:** We omit the proof because it is similar to that of Theorem 3.1.8. □

### 3.2.2 Proofs of the space-time scaling limit theorems

This Section is dedicated to prove the limit theorem stated at the beginning of this section. First, we show some technical results which are needed to prove the desired convergence.

**Lemma 3.2.8** *Let $\mathcal{K}^K(t_1,\varphi_1;t_2,\varphi_2) := \mathrm{Cov}(\langle \varphi_1, M_{t_1}^{2,K}\rangle, \langle \varphi_2, M_{t_2}^{2,K}\rangle)$, where $0 \leq t_2 \leq t_1 < \infty$ and $\varphi_1, \varphi_2 \in S(\mathbb{R}^d)$. Then,*

$$\mathcal{K}^K(t_1,\varphi_1;t_2,\varphi_2) \longrightarrow \mathcal{K}(t_1,\varphi_1;t_2,\varphi_2) \ \text{ as } \ K \longrightarrow \infty,$$

*where*

$$\mathcal{K}(t_1,\varphi_1;t_2,\varphi_2) = \frac{1}{\Gamma(1+\gamma)} \int_0^{t_2} \int_{\mathbb{R}^d} (\mathcal{S}_{t_2-u}\varphi_2)(x)(\mathcal{S}_{t_1-u}\varphi_1)(x)dxd(u^\gamma).$$

**Proof:** From the proof of Theorem 3.1.4 we know that, for $0 \leq t_2 \leq t_1 < \infty$ and $\varphi_1, \varphi_2 \in S(\mathbb{R}^d)$,

$$\mathbb{E}\langle \varphi^K, X_{K^\alpha t}\rangle = K^d \langle \mathcal{S}_t\varphi, \Lambda\rangle = K^d\langle \varphi, \Lambda\rangle.$$

Moreover,

$$\begin{aligned} C_K^2(t_1,\varphi_1;t_2,\varphi_2) &:= \mathrm{Cov}(\langle \varphi_1^K, X_{K^\alpha t_1}\rangle; \langle \varphi_2^K, X_{K^\alpha t_2}\rangle) \\ &= \langle \varphi_2^K \mathcal{S}_{K^\alpha(t_1-t_2)}\varphi_1^K, \Lambda\rangle \\ &\quad + \int_0^{K^\alpha t_2} \int_{\mathbb{R}^d} (\mathcal{S}_{K^\alpha t_2 - r}\varphi_2^K)(x)(\mathcal{S}_{K^\alpha t_1 - r}\varphi_1^K)(x)dx\, dU(r). \end{aligned}$$

Performing the change of variables $u = r/K^\alpha$ and using the self-similarity property of the $\alpha$-stable semigroup, we obtain that

$$\begin{aligned} C_K^2(t_1,\varphi_1;t_2,\varphi_2) &= K^d\langle \varphi_2 \mathcal{S}_{t_1-t_2}\varphi_1, \Lambda\rangle \\ &+K^d\int_0^{t_2}\int_{\mathbb{R}^d}(\mathcal{S}_{t_2-u}\varphi_2)(x)(\mathcal{S}_{t_1-u}\varphi_1)(x)dxdU(K^\alpha u). \quad (3.18)\end{aligned}$$

Note that, by definition

$$\mathcal{K}^K(t_1,\varphi_1;t_2,\varphi_2) = K^{-(d+\alpha\gamma)}C_K^2(t_1,\varphi_1,;t_2,\varphi_2),$$

and from (3.18),

$$\begin{aligned} \mathcal{K}^K(t_1,\varphi_1;t_2,\varphi_2) &= K^{-\alpha\gamma}\langle \varphi_2 \mathcal{S}_{t_1-t_2}\varphi_1, \Lambda\rangle \\ &+K^{-\alpha\gamma}\int_0^{t_2}\int_{\mathbb{R}^d}(\mathcal{S}_{t_2-u})\varphi_2(x)(\mathcal{S}_{t_1-u})\varphi_1(x)dxdU(K^\alpha u). \quad (3.19)\end{aligned}$$

Then, from Lemma (1.7.5) we have that

$$\mathcal{K}^K(t_1,\varphi_1;t_2,\varphi_2) \longrightarrow \mathcal{K}(t_1,\varphi_1;t_2,\varphi_2),$$

as $K \longrightarrow \infty$. □

For the next lemma we recall the notations (2.7) and (2.8), where $m_{B_s}(t,\varphi_1)$ is the mean assuming that the initial particle starts at the random position $B_s$.

**Lemma 3.2.9** *For each $0 \le t_3 \le t_2 \le t_1 < \infty$ and $\varphi_j \in S(\mathbb{R}^d)$, $j = 1,2,3$,*

$$\begin{aligned} \mathbb{E}_x\prod_{j=1}^3\langle\varphi_j, Z_{t_j}\rangle &= \mathbb{E}_x\prod_{j=1}^3\varphi_j(B_{t_j}) \\ &+\int_0^{t_3}\mathbb{E}_x[C_{B_s}(t_3-s,\varphi_3;t_2-s,\varphi_2)m_{B_s}(t_1-s,\varphi_1) \\ &+C_{B_s}(t_3-s,\varphi_3;t_1-s,\varphi_1)m_{B_s}(t_2-s,\varphi_2) \\ &+C_{B_s}(t_2-s,\varphi_2;t_1-s,\varphi_1)m_{B_s}(t_3-s,\varphi_3)]dU(s) \\ &-\mathbb{E}_x\left[\varphi_3(B_{t_3})\int_{t_3}^{t_2}\prod_{j=1}^2 m_{B_s}(t_j-s,\varphi_j)dU(s)\right]. \quad (3.20)\end{aligned}$$

**Proof:** Keeping in mind the notation in Lemma 2.1.2, we have that, for $p = 3$,

$$\mathbb{E}_x \prod_{j=1}^{3} \langle \varphi_j, Z_{t_j} \rangle = \frac{\partial^3}{\partial\theta_3\theta_2\theta_1} Q_{\bar{t}}^3 \theta_{(3)}(x)|_{\theta_j=0},$$

where

$$\begin{aligned}
\frac{\partial^3}{\partial\theta_3\theta_2\theta_1} Q_{\bar{t}}^3 \theta_{(3)}(x) \;=\; & \mathbb{E}_x \Bigg\{ \prod_{j=1}^{3} \varphi_j(B_{t_j}) e^{-\sum_{j=1}^{3} \theta_j \varphi_j(B_{t_j})} \\
& - \int_0^{t_1} \Bigg[ \Psi'''(Q_{\bar{t}-s}^3 \theta_{(3)}(B_s)) \prod_{j=1}^{3} \frac{\partial}{\partial\theta_j} Q_{\bar{t}-s}^3 \theta_{(3)}(B_s) \\
& + \Psi''(Q_{\bar{t}-s}^3 \theta_{(3)}(B_s)) \frac{\partial^2}{\partial\theta_3\partial\theta_2} Q_{\bar{t}-s}^3 \theta_{(3)}(B_s) \frac{\partial}{\partial\theta_1} Q_{\bar{t}-s}^3 \theta_{(3)}(B_s) \\
& + \Psi''(Q_{\bar{t}-s}^3 \theta_{(3)}(B_s)) \frac{\partial^2}{\partial\theta_3\partial\theta_1} Q_{\bar{t}-s}^3 \theta_{(3)}(B_s) \frac{\partial}{\partial\theta_2} Q_{\bar{t}-s}^3 \theta_{(3)}(B_s) \\
& + \Psi''(Q_{\bar{t}-s}^3 \theta_{(3)}(B_s)) \frac{\partial^2}{\partial\theta_2\partial\theta_1} Q_{\bar{t}-s}^3 \theta_{(3)}(B_s) \frac{\partial}{\partial\theta_3} Q_{\bar{t}-s}^3 \theta_{(3)}(B_s) \\
& + \Psi'(Q_{\bar{t}-s}^3 \theta_{(3)}(B_s)) \frac{\partial^3}{\partial\theta_3\partial\theta_2\partial\theta_1} Q_{\bar{t}-s}^3 \theta_{(3)}(B_s) \Bigg] dN_s \\
& + \prod_{j=2}^{3} \varphi_j(B_{t_j}) e^{-\theta_j \varphi_j(B_{t_j})} \int_{t_2}^{t_1} \Psi'(Q_{\bar{t}-s}^1 \theta_{(1)}(B_s)) \frac{\partial}{\partial\theta_1} Q_{\bar{t}-s}^1 \theta_{(1)}(B_s) dN_s \\
& - \varphi_3(B_{t_3}) e^{-\theta_3 \varphi_3(B_{t_3})} \int_{t_3}^{t_2} \Psi'(Q_{\bar{t}-s}^2 \theta_{(2)}(B_s)) \frac{\partial^2}{\partial\theta_2\partial\theta_1} Q_{\bar{t}-s}^2 \theta_{(2)}(B_s) dN_s \\
& - \varphi_3(B_{t_3}) e^{-\theta_3 \varphi_3(B_{t_3})} \int_{t_3}^{t_2} \Psi''(Q_{\bar{t}-s}^2 \theta_{(2)}(B_s)) \prod_{j=1}^{2} \frac{\partial}{\partial\theta_j} Q_{\bar{t}-s}^2 \theta_{(2)}(B_s) dN_s \Bigg\}.
\end{aligned}$$

Since $\Psi(s) = \frac{1}{2}s^2$, we have that

$$\Psi'(0) = 0, \quad \Psi''(0) = 1, \quad \Psi^{(K)}(0) = 0, \quad \text{for} \quad K = 3, 4, \cdots.$$

The proof ends by recalling that $Q_{\bar{t}}^3 \theta_{(3)}(x)|_{\theta_1=\theta_2=\theta_3=0} = 0$. □

**Lemma 3.2.10** *Assume that $d > \alpha\gamma$. For each $0 \le t_3 \le t_2 \le t_1 < \infty$ and $\varphi_j \in S(\mathbb{R}^d)$, $j = 1, 2, 3$,*

$$K^{-(d+\alpha\gamma)3/2} \mathbb{E}_x \prod_{j=1}^{3} \langle \varphi_j^K, Z_{K^\alpha t_j} \rangle \longrightarrow 0 \quad as \quad K \longrightarrow \infty.$$

**Proof:** We start with the first term in the right hand side of (3.20). Using the Markov property of the $\alpha$-stable process, we obtain

$$
\begin{aligned}
&\mathbb{E}_x\left[\prod_{j=1}^{3}\varphi_j^K(B_{K^\alpha t_j})\right] \\
&= \mathbb{E}_x\left\{\mathbb{E}\left[\prod_{j=1}^{3}\varphi_j^K(B_{K^\alpha t_j})\middle|B_{K^\alpha t_3}\right]\right\} \\
&= \mathbb{E}_x\left\{\varphi_3^K(B_{K^\alpha t_3})\mathbb{E}\left[\varphi_1^K(B_{K^\alpha t_1})\varphi_2^K(B_{K^\alpha t_2})|B_{K^\alpha t_3}\right]\right\} \\
&= \int p_{K^\alpha t_3}(x,y)\varphi_3^K(y)\mathbb{E}\left[\varphi_1^K(B_{K^\alpha t_1})\varphi_2^K(B_{K^\alpha t_2})|B_{K^\alpha t_3}=y\right]dy \\
&= \int p_{K^\alpha t_3}(x,y)\varphi_3^K(y)\mathbb{E}_y\left[\varphi_1^K(B_{K^\alpha (t_1-t_3)})\varphi_2^K(B_{K^\alpha (t_2-t_3)})\right]dy \\
&= \int p_{K^\alpha t_3}(x,y)\varphi_3^K(y)\mathbb{E}_y\left\{\mathbb{E}\left[\varphi_1^K(B_{K^\alpha (t_1-t_3)})\varphi_2^K(B_{K^\alpha (t_2-t_3)})|B_{K^\alpha (t_2-t_3)}\right]\right\}dy \\
&= \int p_{K^\alpha t_3}(x,y)\varphi_3^K(y)\int p_{K^\alpha (t_2-t_3)}(y,z)\varphi_2^K(z)\mathbb{E}_z[\varphi_1^K(B_{K^\alpha (t_1-t_2)})]dzdy \\
&= \left(\mathcal{S}_{K\alpha t_3}\left(\varphi_3^K(\cdot)\left(\mathcal{S}_{K^\alpha (t_2-t_3)}\varphi_2^K\right)(\cdot)\left(\mathcal{S}_{K^\alpha (t_1-t_2)}\varphi_1^K\right)(\cdot)\right)\right)(x) \\
&= \left(\mathcal{S}_{t_3}\left(\varphi_3(\cdot)\left(\mathcal{S}_{t_2-t_3}\varphi_2\right)(\cdot)\left(\mathcal{S}_{t_1-t_2}\varphi_1\right)(\cdot)\right)\right)^K(x) \\
&= K^d\left(\mathcal{S}_{t_3}\left(\varphi_3(\cdot)\left(\mathcal{S}_{t_2-t_3}\varphi_2\right)(\cdot)\left(\mathcal{S}_{t_1-t_2}\varphi_1\right)(\cdot)\right)\right)(x).
\end{aligned}
$$

Therefore,

$$
K^{-(d+\alpha\gamma)3/2}\mathbb{E}_x\left[\prod_{j=1}^{3}\varphi_j^K(B_{K^\alpha t_j})\right]\longrightarrow 0,\quad\text{as}\quad K\longrightarrow\infty. \tag{3.21}
$$

Now we deal with the second term in equality (3.20). Namely,

$$
\begin{aligned}
&\int_0^{K^\alpha t_3}\mathbb{E}_x[C_{B_s}(t_3-s,\varphi_3;t_2-s,\varphi_2)m_{B_s}(t_1-s,\varphi_1)]dU(s) \\
&= \int_0^{t_3}\mathbb{E}_x[C_{B_{K^\alpha s}}(K^\alpha(t_3-s),\varphi_3;K^\alpha(t_2-s),\varphi_2)m_{B_{K^\alpha s}}(K^\alpha(t_1-s),\varphi_1)]dU(K^\alpha s) \\
&= \int_0^{t_3}\int_{\mathbb{R}^d}p_{K^\alpha s}(x,y)\left[\left(\varphi_3^K\mathcal{S}_{K^\alpha (t_2-t_3)}\varphi_2^K\right)(y)\right. \\
&\quad\left.+\int_0^{K^\alpha (t_3-s)}\left(\mathcal{S}_{K^\alpha t_3-r}\varphi_3^K\right)(y)\left(\mathcal{S}_{K^\alpha t_2-r}\varphi_2^K\right)(y)dU(r)\left(\mathcal{S}_{K^\alpha (t_1-s)}\varphi_1^K\right)(y)\right]dydU(K^\alpha s),
\end{aligned}
$$

where

$$\begin{aligned}
&\int_0^{t_3}\int_{\mathbb{R}^d} p_{K^\alpha s}(x,y)\varphi_3^K(y)(\mathcal{S}_{K^\alpha(t_2-t_3)}\varphi_2^K)(y)dydU(K^\alpha s)\\
&= \int_0^{t_3}\left(\mathcal{S}_{K^\alpha s}(\varphi_3\mathcal{S}_{t_2-t_3}\varphi_2)^K\right)(x)dU(K^\alpha s)\\
&= \int_0^{t_3}\left(\mathcal{S}_s(\varphi_3\mathcal{S}_{t_2-t_3}\varphi_2)\right)^K(x)dU(K^\alpha s)\\
&= K^d\int_0^{t_3}\left(\mathcal{S}_s(\varphi_3\mathcal{S}_{t_2-t_3}\varphi_2)\right)(x)dU(K^\alpha s)\\
&= O(K^{d+\alpha\gamma}),
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{t_3}\int_{\mathbb{R}^d} p_{K^\alpha s}(x,y)\int_0^{K^\alpha(t_3-s)}\left(\mathcal{S}_{K^\alpha t_3-r}\varphi_3^K\right)(y)\left(\mathcal{S}_{K^\alpha t_2-r}\varphi_2^K\right)(y)dU(r)\\
&\quad\times\left(\mathcal{S}_{K^\alpha(t_1-s)}\varphi_1^K\right)(y)dydU(K^\alpha s)\\
&= \int_0^{t_3}\mathcal{S}_{K^\alpha t_3}\int_0^{t_3-s}\left[\left(\mathcal{S}_{K^\alpha(t_3-h)}\varphi_3^K\right)(\cdot)\left(\mathcal{S}_{K^\alpha(t_2-h)}\varphi_2^K\right)(\cdot)dU(K^\alpha h)\right.\\
&\quad\left.\times\left(\mathcal{S}_{K^\alpha(t_1-s)}\varphi_1^K\right)(\cdot)\right](x)dU(K^\alpha s)\\
&= \int_0^{t_3}\int_0^{t_3-s}\mathcal{S}_{K^\alpha s}\left[\left(\mathcal{S}_{t_3-h}\varphi_3\right)^K(\cdot)\left(\mathcal{S}_{t_2-h}\varphi_2\right)^K(\cdot)\left(\mathcal{S}_{t_1-s}\varphi_1\right)^K(\cdot)\right](x)dU(K^\alpha h)dU(K^\alpha s)\\
&= \int_0^{t_3}\int_0^{t_3-s}\left(\mathcal{S}_s\left[\left(\mathcal{S}_{t_3-h}\varphi_3\right)(\cdot)\left(\mathcal{S}_{t_2-h}\varphi_2\right)(\cdot)\left(\mathcal{S}_{t_1-s}\varphi_1\right)(\cdot)\right](x)\right)^K dU(K^\alpha h)dU(K^\alpha s)\\
&= K^d\int_0^{t_3}\int_0^{t_3-s}\mathcal{S}_s\left[\left(\mathcal{S}_{t_3-h}\varphi_3\right)(\cdot)\left(\mathcal{S}_{t_2-h}\varphi_2\right)(\cdot)\left(\mathcal{S}_{t_1-s}\varphi_1\right)(\cdot)\right](x)dU(K^\alpha h)dU(K^\alpha s)\\
&= O(K^{d+2\alpha\gamma}).
\end{aligned}$$

Therefore,

$$\int_0^{K^\alpha t_3}\mathbb{E}_x[C_{B_s}(t_3-s,\varphi_3;t_2-s,\varphi_2)m_{B_s}(t_1-s,\varphi_1)]dU(s)=O(K^{d+\alpha\gamma})+O(K^{d+2\alpha\gamma}).$$

Similarly, we have that

$$\int_0^{K^\alpha t_3}\mathbb{E}_x[C_{B_s}(t_3-s,\varphi_3;t_1-s,\varphi_1)m_{B_s}(t_2-s,\varphi_2)]dU(s)=O(K^{d+\alpha\gamma})+O(K^{d+2\alpha\gamma}),$$

and

$$\int_0^{K^\alpha t_3}\mathbb{E}_x[C_{B_s}(t_2-s,\varphi_2;t_1-s,\varphi_1)m_{B_s}(t_3-s,\varphi_3)]dU(s)=O(K^{d+\alpha\gamma})+O(K^{d+2\alpha\gamma}).$$

Also, it can be shown as in the preceding calculations that,

$$\mathbb{E}_x\left[\varphi_3(B_{t_3})\int_{t_3}^{t_2}\prod_{j=1}^{2} m_{B_s}(t_j-s,\varphi_j)dU(s)\right] = O(k^{d+\alpha\gamma}).$$

Therefore, putting all these calculations together, we obtain that

$$\mathbb{E}_x\prod_{j=1}^{3}\langle\varphi_j^K, Z_{K^\alpha t_j}\rangle = O(K^d)+O(K^{d+\alpha\gamma})+O(K^{d+2\alpha\gamma})-O(K^{d+\alpha\gamma}).$$

Finally, since $d>\alpha\gamma$,

$$K^{-(d+\alpha\gamma)3/2}\mathbb{E}_x\prod_{j=1}^{3}\langle\varphi_j^K, Z_{K^\alpha t_j}\rangle \longrightarrow 0,$$

as $K\longrightarrow\infty$. □

We are ready to prove convergence of the finite-dimensional distributions of $M^{2,K}$ to those of $M$.

**Proposition 3.2.11** *Assume that $d>\alpha\gamma$. Then, $M^{2,K}\Longrightarrow_f M^{(2)}$ as $K\longrightarrow\infty$.*

**Proof:** Given $0\le t_p\le t_{p-1}\le\cdots\le t_1<\infty$ and $\varphi_1,\cdots,\varphi_p\in S(\mathbb{R}^d)$ we have that, for each $\theta_1,\cdots,\theta_p\in\mathbb{R}$,

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(i\sum_{j=1}^{p}\theta_j\langle\varphi_j, M_{t_j}^{2,K}\rangle\right)\right]\\
=\ &\mathbb{E}\left[\exp\left(i\sum_{j=1}^{p}\theta_j\frac{\langle\varphi_j^K, X_{K^\alpha t_j}\rangle-\mathbb{E}\langle\varphi_j^K, X_{K^\alpha t_j}\rangle}{K^{(d+\alpha\gamma)/2}}\right)\right]\\
=\ &\mathbb{E}\left[\exp\left(i\sum_{j=1}^{p}K^{-(d+\alpha\gamma)/2}\theta_j\langle\varphi_j^K, X_{K^\alpha t_j}\rangle\right)\right]\\
&\times\exp\left(-i\sum_{j=1}^{p}K^{-(d+\alpha\gamma)/2}\theta_j\mathbb{E}\langle\varphi_j^K, X_{K^\alpha t_j}\rangle\right),
\end{aligned}$$

where

$$
\begin{aligned}
& \mathbb{E}\left[\exp\left(i\sum_{j=1}^{p} K^{-(d+\alpha\gamma)/2}\theta_j\langle\varphi_j^K, X_{K^\alpha t_j}\rangle\right)\right] \\
& \exp\left(-\int_{\mathbb{R}^d}\mathbb{E}_x\left[1-e^{i\sum_{j=1}^{p}K^{-(d+\alpha\gamma)/2}\theta_j\langle\varphi_j^K, Z_{K^\alpha t_j}\rangle}\right]dx\right) \\
= \; & \exp\left(\int_{\mathbb{R}^d}\left[\mathbb{E}_x i\sum_{j=1}^{p} K^{-(d+\alpha\gamma)/2}\theta_j\langle\varphi_j^K, Z_{K^\alpha t_j}\rangle - \frac{K^{-(d+\alpha\gamma)}}{2}\mathbb{E}_x\left(\sum_{j=1}^{p}\theta_j\langle\varphi_j^K, Z_{K^\alpha t_j}\rangle\right)^2\right.\right. \\
& \left.\left. - \frac{i}{3!}K^{-(d+\alpha\gamma)3/2}\mathbb{E}_x\left(\sum_{j=1}^{p}\theta_j\langle\varphi_j^K, Z_{K^\alpha t_j}\rangle\right)^3 + \cdots\right]dx\right).
\end{aligned}
$$

Thus, from the preceding calculations and lemmas 3.2.8 and 3.2.10, we get that

$$
\mathbb{E}\left[\exp\left(i\sum_{j=1}^{p}\theta_j\langle\varphi_j, M_{t_j}^{2,K}\rangle\right)\right] = \exp\left(-\frac{1}{2}\sum_{j=1}^{p}\sum_{l=1}^{p}\theta_j\theta_l\mathcal{K}^K(t_j,\varphi_j;t_l,\varphi_l) + o(K^{(d+\alpha\gamma)3/2})\right),
$$

and therefore,

$$
\lim_{K\longrightarrow\infty}\mathbb{E}\left[\exp\left(i\sum_{j=1}^{p}\theta_j\langle\varphi_j, M_{t_j}^{2,K}\rangle\right)\right] = \exp\left(-\frac{1}{2}\sum_{j=1}^{p}\sum_{l=1}^{p}\theta_j\theta_l\mathcal{K}(t_j,\varphi_j;t_l,\varphi_l)\right),
$$

the last relation showing that $M^{2,K} \Longrightarrow_f M^{(2)}$ as $K \longrightarrow \infty$. □

Now we proceed to prove theorems 3.2.1 and 3.2.2, stated at the beginning of this section.

**Proof of Theorem 3.2.1.** Given $t \geq 0$ and $\varphi \in S(\mathbb{R}^d)$ we have that,

$$
\begin{aligned}
\mathbb{E}\left(\frac{\langle\varphi^K, X_{K^\alpha t}\rangle}{K^d} - \langle\varphi,\Lambda\rangle\right)^2 &= K^{-2d}\mathbb{E}\left(\langle\varphi^K, X_{K^\alpha t}\rangle - \mathbb{E}\langle\varphi^K, X_{K^\alpha t}\rangle\right) \\
&= K^{-2d}C_K^2(t,\varphi;t,\varphi).
\end{aligned}
$$

Now, from (3.18) and Lemma 1.7.2 we get that

$$\begin{aligned}
&\lim_{K\longrightarrow\infty} \mathbb{E}\left(\frac{\langle\varphi^K, X_{K^\alpha t}\rangle}{K^d} - \langle\varphi,\Lambda\rangle\right)^2 \\
&= \lim_{K\longrightarrow\infty}\left(K^{-d}\langle\varphi^2,\Lambda\rangle + K^{-d}\int_0^t \langle(\mathcal{S}_{t-s}\varphi)^2,\Lambda\rangle dU(K^\alpha s)\right) \\
&= \lim_{K\longrightarrow\infty}\frac{K^{-d+\alpha\gamma}}{\Gamma(1+\gamma)}\int_0^t \langle(\mathcal{S}_{t-s}\varphi)^2,\Lambda\rangle d(s^\gamma) \\
&= 0
\end{aligned}$$

because of $-d+\alpha\gamma<0$. This ends the proof. □

**Proof of Theorem 3.2.2.** We follow the same approach as in Theorem 3.1.2. We have to verify that conditions (a)-(d) in Theorem 1.8.1 hold. Condition (a) follows in a similar way as to the corresponding condition in Theorem 3.1.2, and condition (b) is exactly Proposition 3.2.11. To show (c) and (d) we will prove that

$$\sup_{K\geq 1}\sup_{0\leq t\leq T}\mathbb{E}\langle\varphi, M_t^{2,K}\rangle^2<\infty,\quad \varphi\in S(\mathbb{R}^d),$$

for each $T>0$. We recall that, as in the proof of Theorem 3.1.2, we can assume that $\varphi>0$. Then, from Lemma 3.2.8 and (3.19) we have that

$$\begin{aligned}
\mathbb{E}\langle\varphi, M_t^{2,K}\rangle^2 &= K^{-\alpha\gamma}\langle\varphi^2,\Lambda\rangle + K^{-\alpha\gamma}\int_0^t\langle(\mathcal{S}_{t-u}\varphi)^2,\Lambda\rangle dU(K^\alpha u) \\
&\leq K^{-\alpha\gamma}\left[\langle\varphi^2,\Lambda\rangle + |\varphi|_p\langle\varphi,\Lambda\rangle\int_0^t dU(K^\alpha u)\right] \\
&\leq K^{-\alpha\gamma}\left[\langle\varphi^2,\Lambda\rangle + |\varphi|_p\langle\varphi,\Lambda\rangle U(K^\alpha t)\right],
\end{aligned}$$

since by (3.12), $\langle(\mathcal{S}_{t-u}\varphi)^2,\Lambda\rangle\leq|\varphi|_p\langle\varphi,\Lambda\rangle$. Then, for each $T>0$

$$\sup_{0\leq t\leq T}\mathbb{E}\langle\varphi, M_t^{2,K}\rangle^2\leq K^{-\alpha\gamma}\left[\langle\varphi^2,\Lambda\rangle + \lfloor\varphi\rfloor_p\langle\varphi,\Lambda\rangle U(K^\alpha T)\right].$$

Hence,

$$\sup_{0\leq t\leq T}\mathbb{E}\langle\varphi, M_t^{2,K}\rangle^2<\infty.$$

□

# Chapter 4

# Occupation time: strong laws of large numbers

Given a *càdlàg* measure-valued process $Y =: \{Y_t, t \geq 0\}$, the *occupation time process* of $Y$ is again a measure-valued process $\{J_t, t \geq 0\}$ defined by

$$\langle \psi, J_t \rangle := \int_0^t \langle \psi, Y_s \rangle ds, \quad t \geq 0,$$

for all bounded measurable function $\psi : \mathbb{R}^d \to \mathbb{R}_+$. Our goal in this Chapter is to show that the occupation time of the branching system $\{X_t, t \geq 0\}$, satisfies a strong law of large numbers similar to that obtained by Cox and Griffeath (1985) and Méléard and Roelly (1992) for the case in which the particle lifetimes are exponentially distributed. See López-Mimbela (1994) for a multi-type version in the Markov case.

Section 1 deals with the strong law of large numbers in the case of heavy-tailed lifetimes. In Section 2 we show the strong law in the case of particle lifetimes with finite mean.

## 4.1 Heavy-tailed life times

Recall that the age-dependent branching particle system with long-living particles is persistent for dimensions $d \geq \alpha\gamma$, $d = \alpha\gamma$ being the critical dimension, see Vatutin and Wakolbinger (1999) and Fleischmann et. al. (2002). We shall prove that the occupation time satisfies a strong law of large numbers in super-critical dimensions $d > \alpha\gamma$. We shall prove the result in two steps. First we show that the result holds for "low" dimensions $\alpha\gamma < d < 2\alpha$; this part of the proof uses the non-Markovian branching system described in Chapter 2. In the second step we consider "large" dimensions $d \geq 2\alpha$, and in this case we use the Markovianized branching system described in Appendix A.

### 4.1.1 Low dimensions

In this section we study the strong law in dimensions $d$ such that $\alpha\gamma < d < 2\alpha$. As we mentioned above, in this case we work directly with the non-Markovian branching system $\{X_t, t \geq 0\}$ described in Chapter 2.

As in Bojdecki et. al. (2004), we define the re-scaled occupation time process, $\{J_T(t) := J_{tT}, t \geq 0\}$, i.e., for any positive bounded measurable function $\varphi$,

$$\begin{aligned} \langle \varphi, J_T(t) \rangle &:= \int_0^{tT} \langle \varphi, X_s \rangle ds \\ &= T \int_0^t \langle \varphi, X_{sT} \rangle ds, \quad t \geq 0. \end{aligned}$$

Notice that, by Fubini's theorem,

$$\mathbb{E}\langle \varphi, J_T(1) \rangle = \langle \varphi, \Lambda \rangle T, \tag{4.1}$$

since $\mathbb{E}\langle \varphi, X_t \rangle = \langle \varphi, \Lambda \rangle$. The main result in this section is the following theorem.

**Theorem 4.1.1** *Assume that $\alpha\gamma < d < 2\alpha$ and that $\varphi$ is a positive test function with*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x)\varphi(y)|x-y|^{\alpha-d} dy dx < \infty$$

*in case that $d < \alpha$. Then, as $T \longrightarrow \infty$,*

$$T^{-1}\langle \varphi, J_T(1)\rangle \stackrel{a.s.}{\rightarrow} \langle \varphi, \lambda\rangle. \tag{4.2}$$

The proof of the theorem is based on the Borel-Cantelli Lemma. In order to use this classical tool we prepare some preliminary results.

Recall that, from (2.11), the covariance functional of the branching system is given by

$$\mathrm{Cov}\left(\langle \varphi, X_s\rangle, \langle \varphi, X_t\rangle\right) = \langle \varphi \mathcal{S}_{t-s}\varphi, \Lambda\rangle + \int_0^s \langle \varphi \mathcal{S}_{t+s-2r}\varphi, \Lambda\rangle dU(r), \quad s \leq t, \quad \varphi \in S(\mathbb{R}^d), \tag{4.3}$$

where $U(r) := \mathbb{E}[N_s] = \sum_{k=0}^{\infty} F^{*k}(r)$.

**Lemma 4.1.2** *Suppose that the hypothesis in Theorem 4.1.1 hold. Then, for each $\epsilon > 0$ and all $T$ large enough,*

$$P\left(|T^{-1}\langle \varphi, J_T(1)\rangle - \langle \varphi, \lambda\rangle| > \epsilon\right) \leq \frac{2}{\epsilon^2}\left(c_3 T^{-2} + c_1 T^{-1} + c_2 T^{-d/\alpha} + c_4 T^{\gamma - d/\alpha}\right),$$

*for some non-negative constants $c_1$, $c_2$, $c_3$ and $c_4$.*

**Proof:** Let $\epsilon > 0$ be given. Then, using Chebyshev's inequality and (4.1),

$$\begin{aligned}
P\left(|T^{-1}\langle \varphi, J_T(1)\rangle - \langle \varphi, \lambda\rangle| > \epsilon\right) &\leq \frac{1}{\epsilon^2}\mathbb{E}\left(T^{-1}\langle \varphi, J_T(1)\rangle - \langle \varphi, \lambda\rangle\right)^2 \\
&= \frac{1}{\epsilon^2 T^2}\mathbb{E}\left(\langle \varphi, J_T(1)\rangle - T\langle \varphi, \lambda\rangle\right)^2 \\
&= \frac{1}{\epsilon^2 T^2}\mathrm{Cov}\left(\langle \varphi, J_T(1)\rangle, \langle \varphi, J_T(1)\rangle\right) \\
&= \frac{1}{\epsilon^2}\int_0^1 \int_0^1 \mathrm{Cov}\left(\langle \varphi, X_{sT}\rangle, \langle \varphi, X_{tT}\rangle\right) dt ds.
\end{aligned}$$

By changing the order of integration we obtain that

$$P\left(|T^{-1}\langle \varphi, J_T(1)\rangle - \langle \varphi, \Lambda\rangle| > \epsilon\right) \leq \frac{2}{\epsilon^2}\int_0^1 dv \int_0^v \mathrm{Cov}\left(\langle \varphi, X_{uT}\rangle, \langle \varphi, X_{vT}\rangle\right) du. \tag{4.4}$$

Therefore, from (4.3) we get that

$$P\left(|T^{-1}\langle \varphi, J_T(1)\rangle - \langle \varphi, \Lambda\rangle| > \epsilon\right) \leq (I) + (II), \tag{4.5}$$

where

$$(I) := \frac{2}{\epsilon^2}\int_0^1 dv \int_0^v du \langle \varphi \mathcal{S}_{T(v-u)}\varphi, \Lambda\rangle,$$

and

$$(II) := \frac{2}{\epsilon^2}\int_0^1 dv \int_0^v du \int_0^u dU(Tr)\langle \varphi \mathcal{S}_{T(v+u-2r)}\varphi, \Lambda\rangle.$$

We proceed to derive upper-bounds for the last two integrals (I) and (II). Firstly, by performing the change of variables $s = (v-u)T$ and $t = vT$, we get that

$$\begin{aligned}\frac{\epsilon^2}{2}(I) &= \frac{1}{T^2}\int_0^T dt \int_0^t ds \langle \varphi \mathcal{S}_s \varphi, \Lambda\rangle \\ &= \frac{1}{T^2}\int_0^A dt \int_0^t ds \langle \varphi \mathcal{S}_s \varphi, \Lambda\rangle + \frac{1}{T^2}\int_A^T dt \int_0^t ds \langle \varphi \mathcal{S}_s \varphi, \Lambda\rangle\end{aligned}$$

for some positive constant $A$, where

$$\begin{aligned}\int_0^t \langle \varphi \mathcal{S}_s \varphi, \Lambda\rangle ds &= \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x) p_s(x-y)\varphi(y) dy\, dx\, ds \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x)\varphi(y) \int_0^t p_s(x-y) ds\, dy\, dx \\ &\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x)\varphi(y) \text{const.} \left(|x-y|^{\alpha-d} + t^{1-d/\alpha}\right) dy\, dx\end{aligned}$$

since

$$\int_0^t p_s(x-y) ds \leq \text{const.} \left(|x-y|^{\alpha-d} + t^{1-d/\alpha}\right)$$

because of self-similarity of the $\alpha$-stable semigroup. Since by assumption

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x)\varphi(y)|x-y|^{\alpha-d} dy\, dx < \infty,$$

we get

$$\begin{aligned}\int_A^T dt \int_0^t \langle \varphi \mathcal{S}_s \varphi, \lambda\rangle ds &\leq \text{const.} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x)\varphi(y) \int_A^T \left(|x-y|^{\alpha-d} + t^{1-d/\alpha}\right) dt\, dy\, dx \\ &= c_1(T-A) + c_2(T^{1-d/\alpha} - A^{1-d/\alpha}),\end{aligned}$$

for some constants $c_1, c_2 > 0$. Therefore,

$$(I) \leq \frac{2}{\epsilon^2}\left(\frac{c_3}{T^2} + c_1\frac{T}{T^2} + c_2\frac{T^{2-d/\alpha}}{T^2}\right), \tag{4.6}$$

where

$$c_3 = \int_0^A dt \int_0^t \langle \varphi \mathcal{S}_s \varphi, \Lambda \rangle ds.$$

In order to estimate the integral $(II)$, we recall that

$$U(t) \sim t^\gamma / \Gamma(1+\gamma) \quad \text{as} \quad t \to \infty$$

due to $1 - F(t) \sim t^{-\gamma}/\Gamma(1-\gamma)$, see Theorem 1.7.5. Then, writing $\hat{\varphi}$ for the Fourier transform of $\varphi$, we obtain

$$\begin{aligned}
\frac{\epsilon^2}{2}(II) &= \int_0^1 dv \int_0^v du \int_0^u dU(Tr) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dy |\hat{\varphi}(y)|^2 e^{-T(v+u-2r)|y|^\alpha} \\
&= \frac{1}{(2\pi)^d} \int_0^1 dv \int_0^v du \int_{\mathbb{R}^d} dy |\hat{\varphi}(y)|^2 \int_0^u dU(Tr) e^{-T(v+u-2r)|y|^\alpha} \\
&\sim \frac{\gamma T^\gamma}{(2\pi)^d} \int_0^1 dv \int_0^v du \int_{\mathbb{R}^d} dy |\hat{\varphi}(y)|^2 \int_0^u e^{-T(v+u-2r)|y|^\alpha} r^{\gamma-1} dr,
\end{aligned}$$

and, after the change of variables $z = (T(v+u-2r))^{1/\alpha} y$, we get that

$$\begin{aligned}
\frac{\epsilon^2}{2}(II) &\sim \frac{\gamma T^\gamma}{(2\pi)^d} \int_0^1 dv \int_0^v du \int_{\mathbb{R}^d} dz \int_0^u T^{-d/\alpha}(v+u-2r)^{-d/\alpha} \\
&\quad \times |\hat{\varphi}(T^{-d/\alpha}(v+u-2r)^{-d/\alpha} z)|^2 e^{-|z|^\alpha} r^{\gamma-1} dr \\
&\leq \frac{\gamma T^{\gamma-d/\alpha} \langle \varphi, \lambda \rangle}{(2\pi)^d} \int_{\mathbb{R}^d} dz e^{-|z|^\alpha} \int_0^1 dv \int_0^v du \int_0^u (v+u-2r)^{-d/\alpha} r^{\gamma-1} dr.
\end{aligned}$$

Changing the order of integration in the expression above yields

$$\begin{aligned}
&= \frac{\gamma T^{\gamma-d/\alpha} \langle \varphi, \lambda \rangle}{(2\pi)^d} \int_{\mathbb{R}^d} dz e^{-|z|^\alpha} \int_0^1 dv \int_0^v r^{\gamma-1} \int_r^v du (u+v-2r)^{-d/\alpha} dr \\
&= \frac{\gamma T^{\gamma-d/\alpha} \langle \varphi, \lambda \rangle}{(2\pi)^d} \int_{\mathbb{R}^d} dz e^{-|z|^\alpha} \int_0^1 dv \int_0^v r^{\gamma-1} \frac{2^{1-d/\alpha}(v-r)^{1-d/\alpha} - (v-r)^{1-d/\alpha}}{1-d/\alpha} dr \\
&= \frac{\gamma T^{\gamma-d/\alpha} \langle \varphi, \lambda \rangle}{(2\pi)^d} \frac{2^{1-d/\alpha}-1}{1-d/\alpha} \int_{\mathbb{R}^d} dz e^{-|z|^\alpha} \int_0^1 dv \int_0^v r^{\gamma-1}(v-r)^{1-d/\alpha} dr.
\end{aligned}$$

Changing again the order of integration we get

$$\begin{aligned}
&= \frac{\gamma T^{\gamma-d/\alpha} \langle \varphi, \lambda \rangle}{(2\pi)^d} \frac{2^{1-d/\alpha}-1}{1-d/\alpha} \int_{\mathbb{R}^d} dz e^{-|z|^\alpha} \int_0^1 r^{\gamma-1} \int_r^1 dv (v-r)^{1-d/\alpha} dr \\
&= \frac{\gamma T^{\gamma-d/\alpha} \langle \varphi, \lambda \rangle}{(2\pi)^d} \frac{2^{1-d/\alpha}-1}{(1-d/\alpha)(2-d/\alpha)} \int_{\mathbb{R}^d} dz e^{-|z|^\alpha} \int_0^1 r^{\gamma-1}(1-r)^{2-d/\alpha} dr,
\end{aligned}$$

where the last equality follows from the assumption $d < 2\alpha$. Hence, for $T$ large enough

$$(II) \leq \frac{2}{\epsilon^2} c_4 T^{\gamma - d/\alpha}. \tag{4.7}$$

Therefore,

$$P\left(|T^{-1}\langle\varphi, J_T(1)\rangle - \langle\varphi, \lambda\rangle| > \epsilon\right) \leq \frac{2}{\epsilon^2}\left(c_3 T^{-2} + c_1 T^{-1} + c_2 T^{-d/\alpha} + c_4 T^{\gamma - d/\alpha}\right).$$

**Proof of Theorem 4.1.1**. Let $\epsilon > 0$ and $a > 1$ be given constants, and let $T_n = a^n$ for $n = 1, 2, \ldots$. Then,

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(|T_n^{-1}\langle\varphi, J_{T_n}(1)\rangle - \langle\varphi, \Lambda\rangle| > \epsilon\right) \\
&\leq \frac{2}{\epsilon^2}\sum_{n=1}^{\infty}\left(c_3 T_n^{-2} + c_1 T_n^{-1} + c_2 T_n^{-d/\alpha} + c_4 T_n^{\gamma - d/\alpha}\right) \\
&= \frac{2}{\epsilon^2}\sum_{n=1}^{\infty}\left(c_3 a^{-2n} + c_1 a^{-n} + c_2 a^{(-d/\alpha)n} + c_4 a^{(\gamma - d/\alpha)n}\right) \\
&< \infty,
\end{aligned}$$

due to the assumption $d > \gamma\alpha$. It follows from the Borel-Cantelli lemma that

$$T_n^{-1}\langle\varphi, J_{T_n}(1)\rangle \stackrel{a.s.}{\to} \langle\varphi, \lambda\rangle, \quad \text{as} \quad n \longrightarrow \infty.$$

Now, we observe that for each $T > 1$, there exists some non-negative integer $n(T)$ such that $a^{n(T)} \leq T \leq a^{n(T)+1}$ and $n(T) \longrightarrow \infty$, as $T \longrightarrow \infty$. Hence,

$$\frac{\langle\varphi, J_{a^{n(T)}}(1)\rangle}{a^{n(T)+1}} \leq \frac{\langle\varphi, J_T(1)\rangle}{T} \leq \frac{\langle\varphi, J_{a^{n(T)+1}}(1)\rangle}{a^{n(T)}},$$

and

$$\frac{\langle\varphi, \lambda\rangle}{a} \leq \liminf_{T\to\infty}\frac{\langle\varphi, J_T(1)\rangle}{T} \leq \limsup_{T\to\infty}\frac{\langle\varphi, J_T(1)\rangle}{T} \leq \langle\varphi, \lambda\rangle a,$$

these inequalities being true for any $a > 1$. Letting $a \to 1$ we get the result. □

**Remark 4.1.3** *Note that the null set in the proof of Theorem 4.1.1 may depend on the test function $\varphi$. A null set can be chosen not to depend on $\varphi$, see proof of Theorem 1 in Iscoe (1986b).*

### 4.1.2 Large dimensions

Our goal in this section is to complete the proof of the strong law of large numbers for the occupation time of our branching system. Namely, we show that Theorem 4.1.1 also holds in large dimensions. To do this, we use the Markovianized branching system $\{\bar{X}_t, t \geq 0\}$ defined in Appendix A. We refer to Appendix A for definitions and notations regarding the process $\{\bar{X}_t, t \geq 0\}$. We shall prove the following theorem.

**Theorem 4.1.4** *Assume that $d \geq 2\alpha$ and that $\bar{X}_0$ is a Poisson random field on $\mathbb{R}_+ \times \mathbb{R}^d$ with intensity measure $F \times \Lambda$. Then, as $t \to \infty$,*

$$\frac{\langle \phi, J_t \rangle}{t} \stackrel{a.s.}{\rightarrow} \langle \phi, \Lambda \rangle,$$

*for all positive test function $\phi$. (Here $F$ also denotes the Lebesgue-Stieltjes measure associated to the lifetime distribution $F$).*

**Remark 4.1.5** *Since we are interested only in the spatial component of our branching system, it is enough to deal with functions which only depend on the spatial coordinate.*

To prove this theorem we will need some preliminary results, starting with

**Proposition 4.1.6** *Assume that $\bar{X}_0$ is as in Theorem 4.1.4. Let $\phi, \psi$ be nonnegative, measurable compactly supported functions from $\mathbb{R}_+ \times \mathbb{R}^d$ to $\mathbb{R}_+$. Then, the joint Laplace functional of the branching particle system and its occupation time is given by*

$$\mathbb{E}\left[e^{-\langle \psi, \bar{X}_t \rangle - \int_0^t \langle \phi, \bar{X}_s \rangle ds}\right] = e^{-\langle V_t^\psi \phi, F \times \Lambda \rangle},$$

*where $V_t^\psi \phi$ satisfies, in the mild sense, the non-linear evolution equation*

$$\begin{aligned} \frac{\partial}{\partial t} V_t^\psi \phi(u,x) &= \mathcal{L} V_t^\psi \phi(u,x) - \lambda(u)[\Phi(1 - V_t^\psi \phi(0,x)) - (1 - V_t^\psi \phi(0,x))] \\ &\quad + \phi(u,x)(1 - V_t^\psi \phi(u,x)), \qquad (4.8) \\ V_0^\psi \phi(u,x) &= 1 - e^{-\psi(u,x)}, \end{aligned}$$

*where $\Phi(s) = \frac{1}{2} + \frac{1}{2}s^2$, for all $s \in [-1,1]$, with $\lambda$ given by (A.1) and $\mathcal{L}$ given by (A.2) in Appendix A.*

**Proof:** First we note that Proposition (A.0.2) also holds for a time-dependent function $\psi$. In particular, we can replace $\psi$ by $W^{\psi}_{T-t}\phi$ for $t \in [0,T]$ with $T > 0$ fixed, where $W^{\psi}_t\phi$ is the mild solution of the partial differential equation

$$\begin{aligned} \frac{\partial}{\partial t} W^{\psi}_t\phi(u,x) &= \mathcal{L}W^{\psi}_t\phi(u,x) + \lambda(u)[\Phi(W^{\psi}_t\phi(0,x)) - W^{\psi}_t\phi(0,x)] - \phi(u,x)W^{\psi}_t\phi(u,x), \\ W^{\psi}_0\phi(u,x) &= e^{-\psi(u,x)}. \end{aligned}$$

We see that, for $T > 0$ fixed, (A.9) can be written as

$$M'_t := e^{\langle \log W^{\psi}_{T-t}\phi, \bar{X}_t\rangle} - \int_0^t \langle \phi, \bar{X}_s\rangle e^{\langle \log W^{\psi}_{T-s}\phi, \bar{X}_s\rangle} ds,$$

hence $M'$ is a martingale on $[0,T]$. Now, applying Corollary 2.3.3 of Ethier and Kurtz (1986) to $M'$, we get that

$$\tilde{M}_t := \exp\left( \langle \log W^{\psi}_{T-t}\phi, \bar{X}_t\rangle - \int_0^t \langle \phi, \bar{X}_s\rangle ds \right), \quad 0 \le t \le T,$$

is a martingale. Then, taking expectations and using the martingale property of $\tilde{M}$ we deduce that

$$\begin{aligned} \mathbb{E}\left[ e^{-\langle \psi, \bar{X}_t\rangle - \int_0^t \langle \phi, \bar{X}_s\rangle ds} \right] &= \mathbb{E}\left[ e^{\langle \log W^{\psi}_t\phi, \bar{X}_0\rangle} \right] \\ &= e^{-\langle 1 - W^{\psi}_t\phi, F\times\Lambda\rangle}, \end{aligned}$$

where the last equality follows from the assumption that the initial population is Poissonian with intensity measure $F \times \Lambda$. Finally, to finish the proof we put $V^{\psi}_t\phi := 1 - W^{\psi}_t\phi$.$\Box$

The next Lemma gives the mean of the occupation time.

**Lemma 4.1.7** *Let $\phi : \mathbb{R}^d \to \mathbb{R}_+$ be a measurable function with compact support. Then, for each $t \ge 0$,*

$$\mathbb{E}\langle \phi, J_t\rangle = \langle \phi, \Lambda\rangle t.$$

**Proof:** For any given function $\phi(x)$, $x \in \mathbb{R}^d$, we define the extended function $\bar{\phi}(u,x) \equiv \phi(x)$, $(u,x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Moreover, for any $k \geq 0$ we define

$$\begin{aligned} L_t(k\bar{\phi}) &:= \mathbb{E}\left[e^{-k\int_0^t \langle \bar{\phi}, \bar{X}_s\rangle ds}\right] \\ &= e^{-\langle V_t(k\bar{\phi}), F\times\Lambda\rangle}, \end{aligned} \tag{4.9}$$

where $V_t(k\bar{\phi})$ satisfies (4.8) with $\bar{\phi}$ substituted by $k\bar{\phi}$, and $\psi \equiv 0$. Notice that

$$\begin{aligned} \mathbb{E}\langle \phi, J_t\rangle &= -\frac{d}{dk}\mathbb{E}\left[\exp -\langle k\bar{\phi}, \mathcal{J}_t\rangle\right]\Big|_{k=0^+} \\ &= \left\langle \frac{d}{dk}V_t^0(k\bar{\phi}), F\times\Lambda\right\rangle \exp\left(-\left\langle V_t^0(k\bar{\phi}), F\times\Lambda\right\rangle\right)\big|_{k=0^+}. \end{aligned}$$

Then, defining $\dot{V}_t\bar{\phi} := \frac{d}{dk}V_t^0(k\bar{\phi})\Big|_{k=0^+}$ and recalling that $V_t^0(0\bar{\phi}) = 0$, we obtain

$$\begin{aligned} \tfrac{\partial}{\partial t}\dot{V}_t\bar{\phi}(u,x) &= \mathcal{L}\dot{V}_t\bar{\phi}(u,x) + \bar{\phi}(u,x) \\ \dot{V}_0\bar{\phi}(u,x) &= 0. \end{aligned}$$

Therefore, recalling that $\{\tilde{T}_t, t \geq 0\}$ denotes the semigroup associated to the process $\{\tilde{\xi}_t, t \geq 0\}$ (see Appendix A Section A.0.1.),

$$\begin{aligned} \dot{V}_t\bar{\phi}(u,x) &= \int_0^t \tilde{T}_{t-s}\bar{\phi}(u,x)ds \\ &= \int_0^t \mathcal{S}_{t-s}\phi(x)ds. \end{aligned}$$

Consequently, using that $\Lambda$ is invariant for the $\alpha$-stable semigroup,

$$\begin{aligned} \mathbb{E}\langle \bar{\phi}, J_t\rangle &= \left\langle \dot{V}_t\bar{\phi}, F\times\Lambda\right\rangle \\ &= \left\langle \int_0^t \mathcal{S}_{t-s}\phi ds, \Lambda\right\rangle \\ &= \int_0^t \langle \phi, \Lambda\rangle\, ds \\ &= \langle \phi, \Lambda\rangle t. \end{aligned}$$

□

The next Lemma provides a bound for the variance of the occupation time. Recall that $\Phi(s) = 1/2 + s^2/2$ for $-1 \le s \le 1$.

**Lemma 4.1.8** *Let $\phi : \mathbb{R}^d \to \mathbb{R}_+$ be a measurable function with compact support. Then, for each $t \ge 0$,*

$$\begin{aligned}\mathrm{Var}\langle \phi, J_t\rangle &\le \Phi''(1)\langle \lambda, F\rangle \mathrm{Const}(\phi)(t + t^{3-d/\alpha}) \\ &\quad + 2\mathrm{Const}(\phi)(t + t^{2-d/\alpha}).\end{aligned} \tag{4.10}$$

**Proof:** Define $\bar{\phi}$ as before. Taking derivative of $V_t(k\bar{\phi})$ with respect to $k$, and using equation (4.8), we obtain

$$\begin{aligned}\frac{\partial^2}{\partial t \partial k} V_t(k\bar{\phi})(u,x) &= A\frac{\partial}{\partial k} V_t(k\bar{\phi})(u,x) + \bar{\phi}(u,x)(1 - V_t(k\bar{\phi})(0,x)) \\ &\quad - k\bar{\phi}(u,x)\frac{\partial}{\partial k} V_t(k\bar{\phi})(u,x) \\ &\quad - \lambda(u)\left[-\Phi'(1 - V_t(k\bar{\phi}(0,x))\frac{\partial}{\partial k} V_t(k\bar{\phi})(0,x) + \frac{\partial}{\partial k} V_t(k\bar{\phi})(0,x)\right]\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^3}{\partial t \partial k^2} V_t(k\bar{\phi})(u,x) &= A\frac{\partial^2}{\partial k^2} V_t(k\bar{\phi})(u,x) - 2\bar{\phi}(u,x)\frac{\partial}{\partial k} V_t(k\bar{\phi})(u,x) \\ &\quad - k\bar{\phi}(u,x)\frac{\partial^2}{\partial k^2} V_t(k\bar{\phi})(u,x) \\ &\quad - \lambda(u)\Bigg[\Phi''\left(1 - V_t(k\bar{\phi})(0,x)\right)\left(\frac{\partial}{\partial k} V_t(k\bar{\phi})(u,x)\right)^2 \\ &\quad - \Phi'(1 - V_t(k\bar{\phi})(0,x))\frac{\partial^2}{\partial k^2} V_t(k\bar{\phi})(0,x) + \frac{\partial^2}{\partial k^2} V_t(k\bar{\phi})(0,x)\Bigg].\end{aligned}$$

Letting $\ddot{V}_t\bar{\phi} = \frac{\partial^2}{\partial K^2} V_t(k\bar{\phi})|_{k=0^+}$ we see that

$$\frac{\partial}{\partial t}\ddot{V}_t\phi(u,x) = \mathcal{L}\ddot{V}_t\bar{\phi}(u,x) - \lambda(u)\Phi''(1)\left(\dot{V}_t\bar{\phi}(0,x)\right)^2 - 2\bar{\phi}(u,x)\ddot{V}_t\bar{\phi}(u,x). \tag{4.11}$$

From Lemma 4.1.7 and (4.11) we obtain

$$\ddot{V}_t\bar{\phi}(u,x) = \int_0^t \tilde{T}_s\left[-\lambda(u)\Phi''(1)\left(\int_0^s \tilde{T}_r\bar{\phi}(u,x)dr\right)^2 - 2\bar{\phi}(u,x)\int_0^s \tilde{T}_r\bar{\phi}(u,x)dr\right]ds.$$

Note that $\mathrm{Var}\langle \phi, J_t\rangle = -\left\langle \ddot{V}_t\bar{\phi}(*,\bullet), F\times\Lambda\right\rangle$. Therefore,

$$\begin{aligned}
\mathrm{Var}\langle\bar{\phi}, J_t\rangle &= \left\langle \int_0^t \tilde{T}_s\left[\lambda(*)\Phi''(1)\left(\int_0^s \tilde{T}_r\bar{\phi}(*,\bullet)dr\right)^2\right.\right.\\
&\quad \left.\left. +2\bar{\phi}(*,\bullet)\int_0^s \tilde{T}_r\bar{\phi}(*,\bullet)dr\right] ds, F\times\Lambda\right\rangle\\
&= \int_0^t \left\langle \lambda(*)\Phi''(1)\left(\int_0^s \tilde{T}_r\bar{\phi}(*,\bullet)dr\right)^2, F\times\Lambda\right\rangle ds\\
&\quad +2\int_0^t \left\langle \bar{\phi}(*,\bullet)\int_0^s \tilde{T}_r\bar{\phi}(*,\bullet)dr, F\times\Lambda\right\rangle ds\\
&=: (A)+(B).
\end{aligned}$$

Notice that, under the choice of $\bar{\phi}$, $\tilde{T}_t\bar{\phi}(u,x) = \mathcal{S}_t\phi(x)$ for all $t\geq 0$, and that $\langle\lambda, F\rangle < \infty$. In fact, using that $\lambda(u)\sim u^{-1}$ and $f(u)\sim u^{-\gamma-1}$, we get that for $A>0$ sufficiently large,

$$\begin{aligned}
\langle\lambda, F\rangle &= \int_0^\infty \lambda(u)f(u)du\\
&= \int_0^A \lambda(u)f(u)du + \int_A^\infty \lambda(u)f(u)du\\
&\sim \int_0^A \lambda(u)f(u)du + \int_A^\infty u^{-1}u^{-\gamma-1}du\\
&< \infty.
\end{aligned}$$

Now,

$$\begin{aligned}
(A) &= \int_0^t \left\langle \lambda(*)\Phi''(1)\left(\int_0^s \mathcal{S}_r\phi(\bullet)dr\right)^2, F\times\Lambda\right\rangle ds\\
&= \langle\lambda, F\rangle\Phi''(1)\int_0^t \left\langle \left(\int_0^s \mathcal{S}_r\phi dr\right)^2, \Lambda\right\rangle ds.
\end{aligned}$$

Moreover, it can be shown that

$$\int_0^t \left\langle \left(\int_0^s \mathcal{S}_r\phi dr\right)^2, \Lambda\right\rangle ds \leq \mathrm{Const}(\phi_2)(t+t^{3-d/\alpha}),$$

and consequently,

$$(A) \leq \Phi''(1)\langle\lambda, F\rangle\mathrm{Const}(\phi_2)(t+t^{3-d/\alpha}).$$

Similarly, for the second term we have that

$$\begin{aligned}(B) &= 2\int_0^t \left\langle \bar{\phi}(*,\bullet)\int_0^s \tilde{T}_r\bar{\phi}(*,\bullet)dr, F\times\Lambda\right\rangle ds\\ &= 2\int_0^t \left\langle \int_0^s \mathcal{S}_r\phi dr, \Lambda\right\rangle ds,\end{aligned}$$

where

$$\int_0^t \left\langle \int_0^s \mathcal{S}_r\phi dr, \Lambda\right\rangle ds \leq \mathrm{Const}(\phi)(t+t^{2-d/\alpha}),$$

hence,

$$(B) \leq 2\mathrm{Const}(\phi)(t+t^{2-d/\alpha}).$$

Finally, combining the bounds for (A) and (B) we get the result. □

**Proof of Theorem 4.1.4**. Applying Chebyshev's inequality we have that for each $t \geq 0$ and $\epsilon > 0$

$$P\left\{\frac{|\langle\phi, J_t\rangle - \langle\phi,\Lambda\rangle|}{t} > \epsilon\right\} \leq \frac{1}{t^2\epsilon^2}\mathrm{Var}\langle\phi, J_t\rangle.$$

Let $a \in (1,\infty)$ and $k_n = a^n$, for $n \in \mathbb{N}$. Then, by Lemma 4.1.8

$$\begin{aligned}&\sum_{n=1}^{\infty} P\left\{\frac{|\langle\phi, J_{k_n}\rangle - \langle\phi,\Lambda\rangle|}{k_n} > \epsilon\right\}\\ &\leq \frac{1}{\epsilon^2}\sum_{n=1}^{\infty}\mathrm{Var}\langle\phi, J_{k_n},\rangle\\ &\leq \frac{1}{\epsilon^2}\Bigg\{\Phi''(1)\langle\lambda, F\rangle\mathrm{Const}(\phi)\sum_{n=1}^{\infty}(k_n + k_n^{3-d/\alpha})/k_n^2\\ &\quad + 2\mathrm{Const}(\phi)\sum_{n=1}^{\infty}(k_n + k_n^{2-d/\alpha})/k_n^2\Bigg\}\\ &\leq \frac{1}{\epsilon^2}\Bigg\{\Phi''(1)\langle\lambda, G^*\rangle\mathrm{Const}(\phi)\sum_{n=1}^{\infty}\left(a^{-n} + a^{(1-d/\alpha)n}\right)\\ &\quad + 2\mathrm{Const}(\phi)\sum_{n=1}^{\infty}\left(a^{-n} + a^{(-d/\alpha)^n}\right)\Bigg\}\\ &< \infty.\end{aligned}$$

Hence, by Borel-Cantelli's Lemma

$$\frac{1}{k_n}\langle \phi, J_{k_n}\rangle \overset{a.s.}{\to} \langle \phi, \Lambda\rangle,$$

as $n \to \infty$.

The proof can be completed in a similar way as was done for Theorem 4.1.1. □

**Remark 4.1.9** *When the particle lifetimes have an exponential distribution with mean $\lambda^{-1}$, Theorem 4.1.4 reduces to Theorem 4 of Méléard and Roelly (1992).*

## 4.2 Life times with general distribution but finite mean

In this Section we assume that the particle lifetimes have a general distribution function with finite mean. Namely, we suppose that $F$ is any non-arithmetic distribution function supported on the non-negative real line with finite mean $\mu > 0$.

In case that the lifetimes have an arbitrary distribution with finite mean, the branching particle system is persistent in dimensions $d > \alpha$, and becomes extinct when $d \leq \alpha$ (see Vatutin and Wakolbinger (1999)). Note that this is exactly the same persistence condition as for the case when the life times are exponentially distributed. The purpose is to show that the strong law of large numbers for the occupation time also holds in dimensions $d \geq 2\alpha$. In fact, we shall show the following theorem.

**Theorem 4.2.1** *Assume that $d \geq 2\alpha$, and that $F$ is non-arithmetic with finite mean $\mu > 0$. Then, for all $\varphi \in S(\mathbb{R}^d)$,*

$$T^{-1}\langle \varphi, J_T(1)\rangle \overset{a.s.}{\to} \langle \varphi, \Lambda\rangle \quad \text{as} \quad T \longrightarrow \infty. \tag{4.12}$$

**Proof:** As in the proof of Lemma 4.1.2, we have that

$$P\left(|T^{-1}\langle\varphi, J_T(1)\rangle - \langle\varphi,\lambda\rangle| > \epsilon\right) \leq \frac{2}{\epsilon^2}\int_0^1 dv \int_0^v \mathrm{Cov}\left(\langle\varphi, X_{uT}\rangle, \langle\varphi, X_{vT}\rangle\right) du. \tag{4.13}$$

Then,

$$P\left(|T^{-1}\langle\varphi, J_T(1)\rangle - \langle\varphi,\Lambda\rangle| > \epsilon\right) \leq (I) + (II), \tag{4.14}$$

where

$$(I) := \frac{2}{\epsilon^2}\int_0^1 dv \int_0^v du \langle\varphi \mathcal{S}_T(v-u)\varphi, \Lambda\rangle,$$

and

$$(II) := \frac{2}{\epsilon^2}\int_0^1\int_0^v\int_0^{Tu} \langle(\mathcal{S}_{Tu-r}\varphi)(\mathcal{S}_{Tv-r}\varphi), \Lambda\rangle dU(r) du\, dv.$$

We recall the bound (4.6) for $(I)$. It remains to upper-bound $(II)$.

Performing the change of variables $h = r/T$ in $(II)$ and using Theorem 1.7.3, we have that, for $T$ large enough,

$$\begin{aligned}
(II) &= \frac{2}{\epsilon^2}\int_0^1\int_0^v\int_0^u \langle(\mathcal{S}_{T(u-h)}\varphi)(\mathcal{S}_{T(v-r)}\varphi), \Lambda\rangle dU(Th) du\, dv \\
&= \frac{2}{\epsilon^2}\int_0^1\int_0^v\int_0^u \langle(\mathcal{S}_{T(u-h)}\varphi)(\mathcal{S}_{T(v-r)}\varphi), \Lambda\rangle d\left[\frac{U(Th)}{Th}Th\right] du\, dv \\
&= \frac{2T}{\epsilon^2\mu}\int_0^1\int_0^v\int_0^u \langle(\mathcal{S}_{T(u-h)}\varphi)(\mathcal{S}_{T(v-r)}\varphi), \Lambda\rangle dh\, du\, dv \\
&= \frac{2T}{\epsilon^2\mu}\int_0^1\int_0^v\int_h^v \langle(\mathcal{S}_{T(u-h)}\varphi)(\mathcal{S}_{T(v-r)}\varphi), \Lambda\rangle du\, dh\, dv,
\end{aligned}$$

where we changed the order of integration to obtain the last equality. It can be seen that, after performing several changes of variables, $(II)$ can be written as

$$\begin{aligned}
(II) &= \frac{2}{\epsilon^2\mu T^2}\int_0^T\int_{\mathbb{R}^d}\int_0^v\int_0^t (\mathcal{S}_s\varphi)(x)(\mathcal{S}_t\varphi)(x) ds\, dt\, dx\, dv \\
&\leq \frac{2}{\epsilon^2\mu T^2}\int_0^T\int_{\mathbb{R}^d}\int_0^v\int_0^v (\mathcal{S}_s\varphi)(x)(\mathcal{S}_t\varphi)(x) ds\, dt\, dx\, dv \\
&= \frac{2}{\epsilon^2\mu T^2}\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(y)\varphi(z)\int_0^v\int_0^v p_{t+s}(y-z) ds\, dt\, dy\, dy\, dv.
\end{aligned}$$

On the other hand, we can show that

$$\int_0^v\int_0^v p_{t+s}(y-z) ds\, dt \leq c\left(|y-z|^{2\alpha-d} + v^{2-d\alpha}\right),$$

for some constant $c > 0$. Hence, for fixed $A > 0$

$$
\begin{aligned}
(II) \quad \leq \quad & \frac{2}{\epsilon^2 \mu T^2} \int_0^A \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(y)\varphi(z) \int_0^v \int_0^v p_{t+s}(y-z) ds\, dt\, dy\, dy\, dv \\
& + \frac{2}{\epsilon^2 \mu T^2} \int_A^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(y)\varphi(z) \int_0^v \int_0^v p_{t+s}(y-z) ds\, dt\, dy\, dy\, dv \\
\leq \quad & \frac{2}{\epsilon^2 \mu T^2} \int_0^A \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(y)\varphi(z) \int_0^v \int_0^v p_{t+s}(y-z) ds\, dt\, dy\, dy\, dv \\
& + \frac{2}{\epsilon^2 \mu T^2} c \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(y)\varphi(z) |y-z|^{2\alpha - d} dy\, dz (T - A) \\
& + \frac{2}{\epsilon^2 \mu T^2} c \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(y)\varphi(z) dy\, dz \frac{(T^{3-d/\alpha} - A^{3-d/\alpha})}{3 - d/\alpha} \\
\leq \quad & \frac{2}{\epsilon^2 \mu} \int_0^A \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(y)\varphi(z) \int_0^v \int_0^v p_{t+s}(y-z) ds\, dt\, dy\, dy\, dv T^{-2} \\
& + \frac{2}{\epsilon^2 \mu} c \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(y)\varphi(z) |y-z|^{2\alpha - d} dy\, dz T^{-1} \\
& + \frac{2}{\epsilon^2 \mu} c \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(y)\varphi(z) dy\, dz \frac{(T^{1-d/\alpha} - A^{3-d/\alpha} T^{-2})}{3 - d/\alpha}.
\end{aligned}
$$

Finally, the proof can be completed in the same way as that of Theorem 4.1.1. □

# Chapter 5

# Occupation time fluctuations: work in progress

In this Chapter we present some preliminary covariance calculations regarding the occupation time fluctuation process of our age-dependent branching system. For the case of exponentially distributed lifetimes see Bojdecki et. al. (2004), Bojdecki et. al. (2006a) and Bojdecki et. al. (2006b).

Throughout this Chapter we assume that $\alpha\gamma < d < 2\alpha$. Notice that, in the case of finite mean lifetimes (which can be regarded as $\gamma = 1$), our assumption on the dimension is the same as in the case of exponentially distributed lifetimes, see Bojdecki et. al. (2006b). When the lifetimes have finite mean, the limit process exhibits long-range dependence. A similar behavior seems to prevail in the case of heavy-tailed lifetimes, although with a different long-range dependence process.

## 5.1 Convergence of covariances

Recall that the occupation time process $\{J_t, t \geq 0\}$ is defined by

$$\langle \varphi, J_t \rangle := \int_0^t \langle \varphi, X_s \rangle ds, \ \ t \geq 0, \ \varphi \in S(\mathbb{R}^d).$$

For each $T > 0$, we introduce the re-scaled occupation time process $J_T(t) := J_{Tt}$, which is given by

$$\langle \varphi, J_T(t) \rangle = \int_0^{Tt} \langle \varphi, X_s \rangle ds = T \int_0^t \langle \varphi, X_{Ts} \rangle ds, \quad t \geq 0,$$

$T$ being a parameter which will tend to infinity.

Notice that, due to criticality and the invariance of $\Lambda$ for the $\alpha-$stable semigroup, we obtain that

$$\mathbb{E}\langle \varphi, J_T(t) \rangle = Tt\langle \varphi, \Lambda \rangle.$$

We define occupation time fluctuation process $\{\mathcal{J}_T(t), t \geq 0\}$ by

$$\begin{aligned} \langle \varphi, \mathcal{J}_T(t) \rangle &= \frac{1}{H_T} \left( \langle \varphi, J_T(t) \rangle - Tt\langle \varphi, \Lambda \rangle \right) \\ &= \frac{T}{H_T} \int_0^t \left( \langle \varphi, X_{Tt} \rangle - \langle \varphi, \Lambda \rangle \right) ds, \quad t \geq 0, \end{aligned}$$

where $H_T$ is certain normalization that we shall choose in such a way that we obtain a non-degenerated limit as $T \to \infty$. From the last identity we can see that, for $0 \leq s \leq t$,

$$\begin{aligned} &\mathrm{Cov}\left( \langle \varphi, \mathcal{J}_T(s) \rangle, \langle \psi, \mathcal{J}_T(t) \rangle \right) \\ &= \frac{T^2}{H_T^2} \int_0^s du \int_0^t dv \mathrm{Cov}\left( \langle \varphi, X_{Tu} \rangle, \langle \psi, X_{Tv} \rangle \right) \\ &= \frac{T^2}{H_T^2} \int_0^s du \int_0^t dv C(uT, \varphi; vT, \psi) \\ &= \frac{T^2}{H_T^2} \left( \int_0^s \int_s^t C(uT, \varphi; vT, \psi) dv\, du \right. \\ &\left. +2 \int_0^s \int_0^v C(uT, \varphi; vT, \psi) du\, dv \right). \end{aligned} \tag{5.1}$$

From Proposition 2.1.5 we have that

$$C(s, \varphi; t, \psi) = \langle \varphi \mathcal{S}_{t-s}\psi, \lambda \rangle + \int_0^s \int_{\mathbb{R}^d} (\mathcal{S}_{s-r}\varphi)(x)(\mathcal{S}_{t-r}\psi)(x) dx dU(r), \tag{5.2}$$

where $U(r) = \sum_{k=0}^{\infty} F^{*k}(r)$.

Now, we express equation (5.2) in terms of Fourier transforms. For any $\varphi \in S(\mathbb{R}^d)$, the Fourier transform $\hat{\varphi}$ of $\varphi$ is defined by

$$\hat{\varphi}(x) = \int_{\mathbb{R}^d} e^{ix\cdot y}\varphi(y)dy,$$

where $x \cdot y$ is the inner product in $\mathbb{R}^d$.

Using Plancherel's formula

$$\langle \varphi\psi, \Lambda\rangle = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{\varphi}(y)\overline{\hat{\psi}}(y)dy,$$

and the fact that

$$\widehat{\mathcal{S}_t\varphi}(x) = e^{-t|x|^\alpha}\hat{\varphi}(x), \tag{5.3}$$

(see Sato (1999)) we get that

$$\begin{aligned}\mathrm{Cov}\left(\langle\varphi, X_s\rangle\langle\psi, X_t\rangle\right) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\hat{\varphi}(y)\overline{\widehat{\mathcal{S}_{t-s}\psi}}(y)dy \\ &\quad + \frac{1}{(2\pi)^d}\int_0^s\int_{\mathbb{R}^d}\left(\widehat{\mathcal{S}_{t-r}\varphi}\right)(y)\left(\overline{\widehat{\mathcal{S}_{s-r}\psi}}\right)(y)dydU(r),\end{aligned}$$

from where we deduce that

$$\mathrm{Cov}\left(\langle\varphi, X_s\rangle\langle\psi, X_t\rangle\right) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\hat{\varphi}(y)\hat{\psi}(y)\left[e^{-(t-s)|y|^\alpha} + \int_0^s e^{-(t+s-2r)|y|^\alpha}dU(r)\right]dy. \tag{5.4}$$

Therefore, substituting (5.4) into (5.1) we see that

$$\begin{aligned}&\mathrm{Cov}\left(\langle\varphi, \mathcal{J}_T(s)\rangle, \langle\psi, \mathcal{J}_T(t)\rangle\right) \\ &= \frac{T^2}{H_T^2(2\pi)^d}\left(\int_0^s\int_s^t\int_{\mathbb{R}^d}\hat{\varphi}(y)\hat{\psi}(y)\left[e^{-T(v-u)|y|^\alpha} + \int_0^u e^{-T(v+u-2r)|y|^\alpha}dU(Tr)\right]dy\,dv\,du\right. \\ &\quad \left. +2\int_0^s\int_0^v\int_{\mathbb{R}^d}\hat{\varphi}(y)\hat{\psi}(y)\left[e^{-T(v-u)|y|^\alpha} + \int_0^u e^{-T(v+u-2r)|y|^\alpha}dU(Tr)\right]dy\,du\,dv\right). \end{aligned} \tag{5.5}$$

Our aim now in to determine the normalizing function $H_T$. To do this, we investigate the asymptotic behavior of (5.5) as $T \to \infty$. Define

$$(I_T) := \frac{T^2}{H_T^2}\int_0^s\int_s^t(2\pi)^{-d}\int_{\mathbb{R}^d}\hat{\varphi}(y)\overline{\hat{\psi}(y)}\int_0^u e^{-T(u+v-2r)|y|^\alpha}dU(Tr)\,dy\,dv\,du,$$

$$(II_T) := 2\frac{T^2}{H_T^2}\int_0^s\int_0^v (2\pi)^{-d}\int_{\mathbb{R}^d}\hat{\varphi}(y)\overline{\hat{\psi}(y)}\int_0^u e^{-T(u+v-2r)|y|^\alpha}dU(Tr)\,dy\,du\,du,$$

and

$$(III_T) := \frac{T^2}{H_T^2}\left(\int_0^s du\int_s^t dv + 2\int_0^s dv\int_0^v du\right)\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\hat{\varphi}(y)\hat{\psi}(y)e^{-T(v-u)|y|^\alpha}dy.$$

Recall that, from Theorem 1.7.5,

$$U(t) \sim \frac{t^\gamma}{\Gamma(1+\gamma)} \quad \text{as} \quad t\to\infty. \tag{5.6}$$

Hence, using (5.6), we get that for $T$ large enough,

$$\begin{aligned}(I_T) &\sim \frac{\gamma T^{2+\gamma}}{(2\pi)^d H_T^2}\int_{\mathbb{R}^d}\int_0^s\int_s^t\int_0^u \hat{\varphi}(y)\overline{\hat{\psi}(y)}e^{-T(u+v-2r)|y|^\alpha}r^{\gamma-1}dr\,dv\,du\,dy\\ &= \frac{\gamma T^{2+\gamma}}{(2\pi)^d H_T^2}\int_{\mathbb{R}^d}\int_0^s\int_0^u \hat{\varphi}(y)\overline{\hat{\psi}(y)}e^{-T(u-2r)|y|^\alpha}r^{\gamma-1}\left[-\frac{e^{-Tv|y|^\alpha}}{T|y|^\alpha}\Big|_{v=s}^t\right]dr\,du\,dy\\ &= \frac{\gamma T^{2+\gamma}}{(2\pi)^d H_T^2}\int_{\mathbb{R}^d}\int_0^s\int_0^u \frac{\hat{\varphi}(y)\overline{\hat{\psi}(y)}}{|y|^\alpha}r^{\gamma-1}\left[e^{-T(s+u-2r)}-e^{-T(t+u-2r)}\right]dr\,du\,dy.\end{aligned}$$

Performing the change of variables $z=(T(s+u-2r))^{1/\alpha}y$ and $z=(T(t+u-2r))^{1/\alpha}y$ we get that

$$\begin{aligned}(I_T) &= \frac{\gamma T^{1+\gamma}}{(2\pi)^d H_T^2}\int_{\mathbb{R}^d}\int_0^s\int_0^u \frac{\hat{\varphi}(T^{-1/\alpha}(s+u-2r)^{-1/\alpha}z)\overline{\hat{\psi}(T^{-1/\alpha}(s+u-2r)^{-1/\alpha}z)}}{T^{-1}(s+u-2r)^{-1}|z|^\alpha}\\ &\quad\times e^{-|z|^\alpha}r^{\gamma-1}T^{-d/\alpha}(s+u-2r)^{-d/\alpha}dr\,du\,dy\\ &\quad-\frac{\gamma T^{1+\gamma}}{(2\pi)^d H_T^2}\int_{\mathbb{R}^d}\int_0^s\int_0^u \frac{\hat{\varphi}(T^{-1/\alpha}(t+u-2r)^{-1/\alpha}z)\overline{\hat{\psi}(T^{-1/\alpha}(t+u-2r)^{-1/\alpha}z)}}{T^{-1}(t+u-2r)^{-1}|z|^\alpha}\\ &\quad\times e^{-|z|^\alpha}r^{\gamma-1}T^{-d/\alpha}(t+u-2r)^{-d/\alpha}dr\,du\,dy.\end{aligned}$$

Hence, putting $H_T = T^{(2+\gamma-d/\alpha)}/2$ and using that $\hat{\varphi}(0)=\langle\varphi,\Lambda\rangle$, we obtain that

$$(I_T)\longrightarrow (I), \quad \text{as} \quad T\longrightarrow\infty,$$

where

$$\begin{aligned}(I) &:= \gamma(2\pi)^{-d}\langle\varphi,\Lambda\rangle\langle\psi,\Lambda\rangle\int_{\mathbb{R}^d}\frac{e^{-|z|^\alpha}}{|z|^\alpha}dz\\ &\quad\times\int_0^s\int_0^u r^{\gamma-1}\left[(s+u-2r)^{1-d/\alpha}-(t+u-2r)^{1-d/\alpha}\right]dr\,du.\end{aligned}$$

Furthermore, changing the order of integration in the last two integrals and using that $d < 2\alpha$, we obtain

$$(I) \quad = \quad \frac{\gamma\langle\varphi,\Lambda\rangle\langle\psi,\Lambda\rangle}{(2\pi)^d(2-d/\alpha)}\int_{\mathbb{R}^d}\frac{e^{-|z|^\alpha}}{|z|^\alpha}dz\left\{\int_0^s r^{\gamma-1}\left[(2s-2r)^{2-d/\alpha}-(s-r)^{2-d/\alpha}\right]dr \right.$$
$$\left. -\int_0^s r^{\gamma-1}\left[(t+s-2r)^{2-d/\alpha}-(t-r)^{2-d/\alpha}\right]dr\right\}. \tag{5.7}$$

Using similar arguments we can show that, with $H_T = T^{(2+\gamma-d/\alpha)}/2$,

$$(II_T) \longrightarrow (II),$$
$$(III_T) \longrightarrow 0,$$

as $T \longrightarrow \infty$, where

$$(II) := \frac{\gamma\langle\varphi,\Lambda\rangle\langle\psi,\Lambda\rangle}{(2\pi)^d(2-d/\alpha)}\int_{\mathbb{R}^d}\frac{e^{-|z|^\alpha}}{|z|^\alpha}dz\left(1-2^{1-d/\alpha}\right)\int_0^s r^{\gamma-1}(s-r)^{2-d/\alpha}dr. \tag{5.8}$$

Therefore, for all $0 \le s \le t$ and $\varphi,\psi \in S(\mathbb{R}^d)$,

$$\lim_{T\longrightarrow\infty}\mathrm{Cov}\left(\langle\varphi,\mathcal{J}_T(s)\rangle,\langle\psi,\mathcal{J}_T(t)\rangle\right)$$
$$= \frac{\gamma\langle\varphi,\Lambda\rangle\langle\psi,\Lambda\rangle}{(2\pi)^d(2-d/\alpha)}\int_{\mathbb{R}^d}\frac{e^{-|z|^\alpha}}{|z|^\alpha}dz\left\{\int_0^s r^{\gamma-1}(s-r)^{2-d/\alpha}dr\right.$$
$$\left. -\int_0^s r^{\gamma-1}(t+s-2r)^{2-d/\alpha}dr+\int_0^s r^{\gamma-1}(t-r)^{2-d/\alpha}dr\right\}. \tag{5.9}$$

Putting

$$\mathcal{Q}_{a,b}(s,t) := \int_0^{s\wedge t} r^a\left[(s\wedge t-r)^b+(s\vee t-2)^b-(t+s-2r)^b\right]dr, \tag{5.10}$$

we see that (5.9) can be written as follows. For all $s,t \ge 0$ and $\varphi,\psi \in S(\mathbb{R}^d)$,

$$\lim_{T\longrightarrow\infty}\mathrm{Cov}\left(\langle\varphi,\mathcal{J}_T(s)\rangle,\langle\psi,\mathcal{J}_T(t)\rangle\right) = \frac{\gamma\langle\varphi,\Lambda\rangle\langle\psi,\Lambda\rangle}{(2\pi)^d(2-d/\alpha)}\int_{\mathbb{R}^d}\frac{e^{-|z|^\alpha}}{|z|^\alpha}dz\,\mathcal{Q}_{\gamma-1,2-d/\alpha}(s,t). \tag{5.11}$$

**Remark 5.1.1** *a) Notice that, although the limit (5.9) does not imply weak convergence of processes, it does imply existence of a Gaussian process with covariance function given* *by (5.10) with* $a = \gamma - 1$ *and* $b = 2 - d/\alpha$.

*b) If we assume a general non-arithmetic finite-mean lifetime distribution, i.e.* $\gamma = 1$, *then, for* $0 \leq s \leq t$, *(5.10) reduces to*

$$\begin{aligned} \mathcal{Q}_{0,2-d/\alpha}(s,t) &= \int_0^s \left[(s-r)^{2-d/\alpha} + (t-2)^{2-d/\alpha} - (t+s-2r)^{2-d/\alpha}\right] dr \\ &= \frac{1}{3-d/\alpha}\left(t^{3-d/\alpha} + s^{3-d/\alpha} - \frac{1}{2}\left[(t+s)^{3-d/\alpha} - (t-s)^{3-d/\alpha}\right]\right) \end{aligned} \quad (5.12)$$

*Note that (5.12) is the covariance function of Theorem 2.2 in Bojdecki et. al. (2006b), which was obtained under the assumptions of exponentially distributed lifetimes and* $\alpha < d < 2\alpha$. *A Gaussian process with covariance function given by (5.12) is called* sub-fractional Brownian motion.

*c) Due to part b) in this remark, we conjecture that Theorem 2.2 in Bojdecki et. al. (2006b) can be extended to a general (non necessarily exponential) lifetime distribution with finite mean.*

*d) In case of heavy-tailed lifetime we conjecture that a result similar to Theorem 2.2 in Bojdecki et. al. (2006b) should hold, with a different long-range dependence self-similar process.*

**Other problems that remain to be investigated on their own right.**

1) Under what conditions on the parameters, $a$ y $b$, (5.10) is a covariances function?

2) Investigate long-range dependence of the process with covariance function (5.10), which could be called *weighted sub-fractional Brownian motion (w-subfbm)*, see Bojdecki et. al. (2007a).

# Appendix A

# Markovianizing the branching system

### A.0.1 The basic process

In this section we consider the $\mathbb{R}_+ \times \mathbb{R}^d$-valued Markov process $\tilde{\xi}$ defined by $\tilde{\xi} := \{(\eta(t), \xi(t)), t \geq 0\}$, where $\xi \equiv \{\xi(t),\ t \geq 0\}$ is the spherically symmetric $\alpha$-stable motion, and the process $\eta \equiv \{\eta_t, t \geq 0\}$, defined below, represents the age at time $t \geq 0$ in a renewal process with arrival distribution function $F$.

The process $\tilde{\xi}$ models the evolution of a population starting at time $t = 0$ with an individual of age $\eta_0 \in [0, \infty)$ at position $\xi_0 \in \mathbb{R}^d$. During its remaining (or "residual") lifetime, this individual develops a random motion $\tilde{\xi}$ in $\mathbb{R}_+ \times \mathbb{R}^d$ whose position component follows an $\alpha$-stable motion. At death, the individual is replaced by a particle of age zero at the place where the parent individual died, and the new particle evolves in the same way as its progenitor, and so on.

Let $\{\tau_i, i = 1, 2, \ldots\}$ be a sequence of i.i.d. random variables with a common distribution function $F$ which has continuous density $f$, and such that $F(0) = 0$ and $F(t) < 1$ for all $t \in \mathbb{R}_+$. Let $u \geq 0$ be a given number, and let $\tau_0^u$ be an independent random variable with distribution function $F_u$ defined by

$$F_u(t) := P\{\tau_0^u \le t\} = \frac{F(u+t) - F(u)}{1 - F(u)}.$$

The age-process (starting at $u$) is defined as follows:

$$\begin{aligned}
\eta_t &= u + t \quad \text{for} \quad 0 \le t < \tau_0^u, \\
\eta_t &= t - \tau_0^u \quad \text{for} \quad \tau_0^u \le t < \tau_0^u + \tau_1, \\
&\vdots \\
\eta_t &= t - (\tau_0^u + \tau_1 + \cdots + \tau_n) \quad \text{for} \quad \tau_0^u + \tau_1 + \cdots + \tau_n \le t < \tau_0^u + \tau_1 + \cdots + \tau_{n+1}, \\
& n = 1, 2, \ldots.
\end{aligned}$$

The process $\{\eta_t, t \ge 0\}$ is Markovian, and is known as the age process, see Joffe (1992). Notice that $\{\eta_t, t \ge 0\}$ can be seen as a piecewise deterministic Markov process (see e.g. Rolski et. al. (1999), Chapter 11) whose first jump time $\delta_0$ has distribution

$$P_u\{\delta_0 > t\} = e^{-\int_u^{u+t} \lambda(s) ds}, \ \ t \ge 0,$$

where

$$\lambda(u) := \frac{f(u)}{1 - F(u)}, \ \ u \ge 0, \tag{A.1}$$

and $P_u$ means that $\eta_0 = u$, and the successive jumps $\{\delta_1, \delta_1, \cdots\}$ have distribution function given by

$$P_u\{\delta_k > t\} = e^{-\int_0^t \lambda(s) ds}, \quad t \ge 0, \quad k = 1, 2, \ldots.$$

Let $\tilde{S} \equiv \{\tilde{S}_t, \, t \ge 0\}$ be the semigroup of linear operators corresponding to $\tilde{\xi}$. Our aim here is to find the infinitesimal generator $\mathcal{L}$ of $\tilde{S}$.

**Proposition A.0.1** *Let $\psi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ be a bounded function such that $\psi(\cdot, x) \in C_b^1(\mathbb{R}_+)$ for any $x \in \mathbb{R}^d$, and $\psi(u, \cdot) \in \mathrm{Dom}(\Delta_\alpha) = C_c^\infty(\mathbb{R}^d)$ for any $u \in \mathbb{R}_+$. Then,*

$$\mathcal{L}\psi(u,x) = \frac{\partial \psi(u,x)}{\partial u} + \Delta_\alpha \psi(u,x) - \lambda(u)[\psi(u,x) - \psi(0,x)]. \tag{A.2}$$

*Hence, $\{\tilde{\xi}_t, t \ge 0\}$ is a homogeneous Markov process whose infinitesimal generator $\mathcal{L}\psi$, $\psi \in \mathrm{Dom}(\mathcal{L})$, is given by the right hand side of (A.2).*

**Proof:** For every $t \geq 0$ we need to find the limit

$$\mathcal{L}(t)\psi(u,x) = \lim_{h\downarrow 0} h^{-1}\mathbb{E}\left[\psi(\tilde{\xi}_{t+h}) - \psi(\tilde{\xi}_t)|\tilde{\xi}_t = (u,x)\right].$$

For any $h \geq 0$ we have that

$$\begin{aligned}
&h^{-1}\mathbb{E}\left[\psi(\tilde{\xi}_{t+h}) - \psi(u,x)|\tilde{\xi}_t = (u,x)\right] \\
&= h^{-1}\mathbb{E}\left[\psi(\tilde{\xi}_{t+h})|\tilde{\xi}_t = (u,x)\right] - h^{-1}\psi(u,x) \\
&= h^{-1}\left\{e^{-\int_u^{u+h}\lambda(s)ds}\mathcal{S}_h^\alpha\psi(u+h,.)(x) + \int_0^h \lambda(u+r)e^{-\int_u^{u+r}\lambda(s)ds}\right. \\
&\quad \left.\times \int_{\mathbb{R}^d} p_r^\alpha(x,y)dydr\mathbb{E}[\psi(\tilde{\xi}_{t+h}) \mid \tilde{\xi}_{t+r} = (0,y)]\right\} - h^{-1}\psi(u,x) \\
&= \frac{1}{h}e^{-\int_u^{u+h}\lambda(s)ds}\left[\mathcal{S}_h^\alpha\psi(u+h,.)(x) - \mathcal{S}_h^\alpha\psi(u,.)(x)\right] \\
&\quad +\frac{1}{h}e^{-\int_u^{u+h}\lambda(s)ds}\left[\mathcal{S}_h^\alpha\psi(u,.)(x) - \psi(u,.)(x)\right] + \frac{1}{h}\left(e^{-\int_u^{u+h}\lambda(s)ds} - 1\right)\psi(u,x) \\
&\quad +\frac{1}{h}\int_0^h \lambda(u+r)e^{-\int_u^{u+r}\lambda(s)ds}\mathcal{S}_r^\alpha\mathbb{E}[\psi(\tilde{\xi}_{t+h}) \mid \tilde{\xi}_{t+r} = (0,.)](x)dr.
\end{aligned}$$

Therefore, letting $h \downarrow 0$ and using the strong continuity of $\mathcal{S}$ in the first term on the last equality, we get (A.2).

### A.0.2 Markovianizing an age-dependent branching particle system

Let $X \equiv \{X_t, t \geq 0\}$ be the branching system defined in the introduction. For any $t \geq 0$, let $\bar{X}_t$ denote the population in $\mathbb{R} \times \mathbb{R}^d$ obtained by attaching to each individual $\delta_x \in X_t$ its age. Namely, for each $t \geq 0$

$$\bar{X}_t = \sum_i \delta_{(\eta_t^i, \xi_t^i)},$$

where $\eta_t^i$ and $\xi_t^i$ denotes the age and position, respectively, of the $i^{th}$ particle at time $t$ and the summation is over all particles alive at time $t$. Notice that all newborns get age 0. In this Section we assume that the branching law is critical, i.e, a random variable $\zeta$ with $p_k := P(\zeta = k)$, $k = 0, 1, 2, \cdots$, such that $\sum_{k=1}^\infty kp_k = 1$. Let $\Phi$ be the probability generating function of $\zeta$, i.e., $\Phi(s) := \mathbb{E}\, s^{\zeta}$, $|s| \leq 1$.Thoroughout this section we suppose that the initial state $\bar{X}_0$ is a general locally finite counting measure on $\mathbb{R} \times \mathbb{R}^d$.

Let $\bar{X} \equiv \{\bar{X}_t,\, t \geq 0\}$. In this section we calculate the infinitesimal generator $\mathcal{G}$ of $\bar{X}$ on certain cylindrical functions $G_\psi \in \mathrm{Dom}(\mathcal{G})$. Namely, let $\psi : \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow (0, 1]$ be a function having compact support, and such that $\psi \in \mathrm{Dom}(\mathcal{L})$. Let $\nu$ be a locally finite counting measure on $\mathbb{R} \times \mathbb{R}^d$. Define

$$G_\psi(\nu) := \exp\left(\langle \log \psi, \nu \rangle\right), \tag{A.3}$$

sometimes we write (A.3) as

$$G_\psi(\nu) := \exp\left(\langle \log \psi(*, \cdot), \nu \rangle\right), \tag{A.4}$$

to emphasize that $*$ is a variable in $\mathbb{R}_+$ and $\cdot$ is a variable in $\mathbb{R}^d$. The infinitesimal generator of $\{\bar{X}_t; t \geq 0\}$, evaluated at the function $G_\psi(\nu)$, is defined by

$$\mathcal{G}G_\psi(\nu) := \lim_{t \to 0} t^{-1} \mathbb{E}\left[G_\psi(\bar{X}_t) - G_\psi(\nu) \mid \bar{X}_0 = \nu\right].$$

Following Ikeda et. al. (1969) we can see that $\mathcal{G}$ can be expressed as

$$\mathcal{G}G_\psi(\nu) = \mathcal{B}G_\psi(\nu) + \mathcal{D}G_\psi(\nu),$$

where $\mathcal{B}$ and $\mathcal{D}$ are the infinitesimal generators corresponding to the branching and diffusion parts, respectively. We first evaluate the branching part. Assume that the system starts with a finite population $\bar{X}_0 = \sum_{k=1}^n \delta_{(u_k, x_k)}$. Let $\tau_*$ be the time to the first branching, i.e.

$$\tau_* = \min\{\tau_1 - u_1, \cdots, \tau_n - u_n\},$$

where $\tau_i$ is the total life time of the $i$-th particle, $i = 1, \ldots, n$. Then

$$\begin{aligned}\mathcal{B}G_g\left(\bar{X}_0\right) &= \lim_{h\downarrow 0} h^{-1}\mathbb{E}\left[G_g(\bar{X}_h) - G_g(\bar{X}_0)|\bar{X}_0 = \sum_{k=1}^{n}\delta_{(u_k,x_k)}\right] \\ &= \lim_{h\downarrow 0} h^{-1}\sum_{i=1}^{n} P\left\{0 < \tau_i - u_i \le h,\right. \\ &\quad \left.\bigwedge_{j=1,j\neq i}^{n}(\tau_j - u_j) > \tau_i - u_i|\tau_1 > u_1, \cdots, \tau_n > u_n\right\} \\ &\quad \times\sum_{k=0}^{\infty} p_k\left[G_g(X_0 - \delta_{(u_i,x_i)} + k\delta_{(0,x_i)}) - G_g(X_0)\right],\end{aligned}$$

where to obtain the second equality we used that

$$\begin{aligned}\{\tau_* \le h|\tau_1 > u_1, \cdots, \tau_n > u_n\} &= \bigcup_{i=1}^{n}\{0 < \tau_i - u_i \le h, \\ &\quad \bigwedge_{j=1,j\neq i}^{n}(\tau_j - u_j) > \tau_i - u_i|\tau_1 > u_1, \cdots, \tau_n > u_n\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&P\{\tau_* \le h|\tau_1 > u_1, \cdots, \tau_n > u_n\} \\ &= P\left\{0 < \tau_i - u_i \le h, \bigwedge_{j=1,j\neq i}^{n}(\tau_j - u_j) > \tau_i - u_i|\tau_1 > u_1, \cdots, \tau_n > u_n\right\} \\ &= \sum_{i=1}^{n}\frac{P\left\{0 < \tau_i - u_i \le h, \bigwedge_{j=1,j\neq i}^{n}(\tau_j - u_j) > \tau_i - u_i, \tau_1 > u_1, \cdots, \tau_n > u_n\right\}}{P\left\{\tau_1 > u_1, \cdots, \tau_n > u_n\right\}} \\ &= \frac{\sum_{i=1}^{n}\int_{u_i}^{u_i+h}\prod_{j=1,j\neq i}^{n} P\{\tau_j - u_j > r - u_i, \tau_j > u_j\}f_{\tau_i}(r)dr}{\prod_{i=1}^{n} P\{\tau_i > u_i\}} \\ &= \frac{\sum_{i=1}^{n}\int_{u_i}^{u_i+h}\prod_{j=1,j\neq i}^{n} e^{-\int_0^{(r+u_j-u_i)\wedge u_j}\lambda(s)ds}\lambda(r)e^{-\int_0^r\lambda(s)ds}dr}{\prod_{i=1}^{n} P\{\tau_i > u_i\}},\end{aligned}$$

and so,

$$\lim_{h\downarrow 0} h^{-1}P\{\tau_* \le h|\tau_1 > u_1, \cdots, \tau_n > u_n\} = \sum_{i=1}^{n}\lambda(u_i).$$

Consequently,

$$\begin{aligned}\mathcal{B}G_g\left(\bar{X}_0\right) &= \sum_{i=1}^{n}\lambda(u_i)\sum_{k=0}^{\infty}p_k\left[G_g(\bar{X}_0-\delta_{(u_i,x_i)}+k\delta_{(0,x_i)})-G_g(\bar{X}_0)\right]\\ &= \langle\lambda(*)\sum_{k=0}^{\infty}p_k\left[G_g(\mu-\delta_{(*,\bullet)}+k\delta_{(0,\bullet)})-G_g(\mu)\right],\bar{X}_0\rangle.\end{aligned}$$

Hence, for all $\nu\in\mathcal{N}$,

$$\begin{aligned}\mathcal{B}G_\psi(\nu) &= \langle\lambda(*,\cdot)\sum_{k=0}^{\infty}p_k\left[G_\psi(\nu-\delta_{(*,\cdot)}+k\delta_{(0,\cdot)})-G_\psi(\nu)\right],\nu\rangle\\ &= \langle\lambda(*)\sum_{k=0}^{\infty}p_k\left[\frac{\psi^k(0,\cdot)}{\psi(*,\cdot)}-1\right],\nu\rangle\\ &= G_\psi(\nu)\langle\lambda(*)\frac{\Phi(\psi(0,\cdot))-\psi(*,\cdot)}{\psi(*,\cdot)},\nu\rangle. \end{aligned}\tag{A.5}$$

Let us see what the diffusion part is. Again, we suppose that $\bar{X}_0=\sum_{k=1}^{n}\delta_{(u_k,x_k)}$. Then,

$$\begin{aligned}&\mathbb{E}[G_\psi(\bar{X}_h)\mid\bar{X}_0,\tau_*>h]P(\tau_*>h\mid\tau_1>u_1,\cdots,\tau_n>u_n)\\ &= \sum_{k=1}^{n}e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\mathcal{S}_h\psi(u_k+h,x_k)\prod_{j=1,j\neq k}^{n}e^{-\int_{u_j}^{u_j+h}\lambda(r)dr}\mathcal{S}_h\psi(u_j+h,x_j).\end{aligned}$$

Using independence we obtain that

$$\mathbb{E}[G_\psi(\bar{X}_h)-G_\psi(\bar{X}_0)\mid\bar{X}_0]=\prod_{k=1}^{n}e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\mathcal{S}_h\psi(u_k+h,x_k),$$

hence,

$$\begin{aligned}
\mathcal{D}G_\psi(\bar{X}_0) &:= \lim_{h\downarrow 0} h^{-1}\mathbb{E}[G_\psi(\bar{X}_h) - G_\psi(\bar{X}_0) \mid \bar{X}_0] \\
&= \lim_{h\downarrow 0} h^{-1}\left[\prod_{k=1}^{n} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\psi(u_k+h,x_k) - \prod_{k=1}^{n}\mathcal{S}_h\psi(u_k,x_k)\right] \\
&= \lim_{h\downarrow 0} h^{-1}\left[\prod_{k=1}^{n} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\mathcal{S}_h\psi(u_k+h,x_k) - \prod_{k=1}^{n} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\psi(u_k+h,x_k)\right. \\
&\quad + \prod_{k=1}^{n} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\psi(u_k+h,x_k) - \prod_{k=1}^{n} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\psi(u_k,x_k) \\
&\quad \left. + \prod_{k=1}^{n} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\psi(u_k,x_k) - \prod_{k=1}^{n}\psi(u_k,x_k)\right] \\
&= \sum_{k=1}^{n}\left[\Delta_\alpha\psi(u_k,x_k) + \frac{\partial}{\partial u}\psi(u_k,x_k) - \lambda(u_k)\psi(u_k,x_k)\right]\prod_{j=1,j\neq k}^{n}\psi(u_j,x_j). \quad \text{(A.6)}
\end{aligned}$$

We will prove the last equality only for $n = 2$; for general $n$ the calculations are similar but more involved. Take $n = 2$ and define

$$(I) := \lim_{h\downarrow 0} h^{-1}\left[\prod_{k=1}^{2} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\mathcal{S}_h\psi(u_k+h,x_k) - \prod_{k=1}^{2} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\psi(u_k+h,x_k)\right],$$

$$(II) := \lim_{h\downarrow 0} h^{-1}\left[\prod_{k=1}^{2} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\psi(u_k+h,x_k) - \prod_{k=1}^{2} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\psi(u_k,x_k)\right],$$

and

$$(III) := \lim_{h\downarrow 0} h^{-1}\left[\prod_{k=1}^{2} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\psi(u_k,x_k) - \prod_{k=1}^{2}\psi(u_k,x_k)\right].$$

Applying the chain rule to the third term we get that

$$(III) = -\sum_{k=0}^{2}\lambda(u_k)\prod_{j=1,j\neq k}^{2}\psi(u_j,x_j).$$

For the second term,

$$\begin{aligned}(II) &= \lim_{h\downarrow 0}\left[\prod_{k=1}^{2} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\psi(u_k+h,x_k) - \prod_{k=1}^{2} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\psi(u_k,x_k)\right]\\ &= \lim_{h\downarrow 0} h^{-1}\Big\{e^{-\int_{u_1}^{u_1+h}\lambda(r)dr}[\psi(u_1+h,x_1)-\psi(u_1,x_1)]e^{-\int_{u_2}^{u_2+h}\lambda(r)dr}\psi(u_2,x_2)\\ &\quad + e^{-\int_{u_2}^{u_2+h}\lambda(r)dr}[\psi(u_2+h,x_2)-\psi(u_2,x_2)]e^{-\int_{u_1}^{u_1+h}\lambda(r)dr}\psi(u_1,x_1)\Big\}\\ &= \psi(u_2,x_2)\frac{\partial}{\partial u}\psi(u_1,x_1)+\psi(u_1,x_1)\frac{\partial}{\partial u}\psi(u_2,x_2).\end{aligned}$$

Finally, we evaluate the first term

$$\begin{aligned}(I) &= \lim_{h\downarrow 0} h^{-1}\left[\prod_{k=1}^{2} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\mathcal{S}_h\psi(u_k+h,x_k) - \prod_{k=1}^{2} e^{-\int_{u_k}^{u_k+h}\lambda(r)dr}\phi(u_k+h,x_k)\right]\\ &= \lim_{h\downarrow 0} h^{-1}\Big\{e^{-\int_{u_1}^{u_1+h}\lambda(r)dr}[\mathcal{S}_h\psi(u_1+h,x_1)-\psi(u_1,x_1)]e^{-\int_{u_2}^{u_2+h}\lambda(r)dr}\mathcal{S}_h\psi(u_2,x_2)\\ &\quad e^{-\int_{u_2}^{u_2+h}\lambda(r)dr}[\mathcal{S}T_h\psi(u_2+h,x_2)-\psi(u_2,x_2)]e^{-\int_{u_1}^{u_1+h}\lambda(r)dr}\mathcal{S}_h\psi(u_1,x_1)\Big\}\\ &= \psi(u_2,x_2)\Delta_\alpha\psi(u_1,x_1)+\psi(u_1,x_1)\Delta_\alpha\psi(u_2,x_2).\end{aligned}$$

Putting together the expression for $(I)$, $(II)$ and $(III)$ yields (A.6) for $n=2$.

Note that (A.6) can be written as

$$\mathcal{D}G_\psi(\nu) = G_\psi(\nu)\left\langle \frac{\Delta_\alpha\psi(*,\cdot)+\frac{\partial}{\partial *}\psi(*,\cdot)-\lambda(*)\psi(*,\cdot)}{\psi(*,\cdot)},\nu\right\rangle, \tag{A.7}$$

hence, adding (A.5) and (A.7) we get that

$$\mathcal{G}G_\psi(\nu) = G_\psi(\nu)\left\langle \frac{\mathcal{L}\psi(*,\cdot)+\lambda(*)[\Phi(\psi(0,\cdot))-\psi(0,\cdot)]}{\psi(*,\cdot)},\nu\right\rangle. \tag{A.8}$$

Using the Martingale problem for $\{\bar{X}_t, t\geq 0\}$ we can summarize the previous calculations in the following proposition, see Ethier and Kurtz (1986) Chapter 4.

**Proposition A.0.2** *Let $\{\bar{X}_t, t\geq 0\}$ be the branching particle system defined at the beginning of this Section. Then for each $\psi\in Dom(\mathcal{L})$ such that $0<\|\psi\|\leq 1$ and $\psi>0$, the process*

$$M_t := e^{\langle \log\psi,\bar{X}_t\rangle} - \int_0^t e^{\langle\log\psi,\bar{X}_s\rangle}\left\langle \frac{\mathcal{L}\psi(*,\cdot)+\lambda(*)[\Phi(\psi(0,\cdot))-\psi(0,\cdot)]}{\psi(*,\cdot)},\bar{X}_s\right\rangle ds, \tag{A.9}$$

*is a martingale.*

**Remark A.0.3** *If* $1-F(t)=e^{-\lambda t}$ *for some constant* $\lambda>0$*, then from (A.1) we get that* $\lambda(t)\equiv\lambda$*. Hence, taking* $\psi(u,x)\equiv\psi(x)$*, the martingale (A.9) in Proposition A.0.2 can be written as*

$$M_t = e^{\langle \log\psi, X_t\rangle} - \int_0^t e^{\langle \log\psi, X_s\rangle} \left\langle \frac{\Delta_\alpha \psi(\cdot) + \lambda[\Phi(\psi(\cdot)) - \psi(\cdot)]}{\psi(\cdot)}, X_s \right\rangle ds, \tag{A.10}$$

*which coincides with the previous known results, see for example Méléard and Roelly (1992).*

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